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**Multicontact mappings  
on Hessenberg manifolds**

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## Contents

Introduction	v
Chapter 1. Preliminaries	1
1. Simple Lie algebras	1
2. Multicontact mappings on $G/P$	3
3. A case study: $\mathfrak{sl}(3, \mathbb{R})$	8
Chapter 2. Polynomial basis for split simple Lie algebras	11
1. The polynomial algebra $\mathcal{P}$	11
2. The split case	11
Chapter 3. Hessenberg manifolds	27
1. The basic context: the Hessenberg flags	27
2. Real Hessenberg manifolds	29
Chapter 4. Multicontact vector fields on Hessenberg manifolds	35
1. A multicontact structure on Hessenberg manifolds	35
2. A set of multicontact vector fields	46
3. Iwasawa sub-models	56
4. A counter-example	70
Bibliography	77



## Introduction

This thesis is concerned with the study of multicontact structures on Hessenberg manifolds and of the mappings that preserve them. The setting in which our considerations take place is that of parabolic geometry, namely a homogeneous space  $G/P$  where  $G$  is a semisimple Lie group and  $P$  is a parabolic subgroup of  $G$ . Furthermore, it is assumed that  $G$  has real rank greater than one and that  $P$  is minimal. We show that it is possible to define a notion of multicontact mapping, hence of multicontact vector field, on every Hessenberg submanifold  $\text{Hess}_{\mathcal{R}}(H)$  of  $G/P$  associated to a regular element  $H$  in the Cartan subspace  $\mathfrak{a}$  of the Lie algebra  $\mathfrak{g}$  of  $G$ . The Hessenberg combinatorial data, namely the subset  $\mathcal{R}$  of the positive restricted roots  $\Sigma_+$  relative to  $(\mathfrak{g}, \mathfrak{a})$  that defines the type of the manifold, single out an ideal  $\mathfrak{n}_{\mathcal{C}}$  in the nilpotent Iwasawa subalgebra of  $\mathfrak{g}$ , labeled by the complement  $\mathcal{C} = \Sigma_+ \setminus \mathcal{R}$ . By means of a reduction theorem, it is shown that without loss of generality one can work under the assumption that  $\mathcal{R}$  contains all the simple restricted roots. In order to avoid certain degeneracies, we assume further that  $\mathcal{R}$  contains all height-two restricted roots as well. We prove that the normalizer of  $\mathfrak{n}_{\mathcal{C}}$  in  $\mathfrak{g}$  modulo  $\mathfrak{n}_{\mathcal{C}}$  is naturally embedded in the Lie algebra of multicontact vector fields on  $\text{Hess}_{\mathcal{R}}(H)$ . If the data  $\mathcal{R}$  satisfy the property of encoding a finite number of positive root systems, each corresponding to an Iwasawa nilpotent algebra, then the above quotient actually coincides with the Lie algebra of multicontact vector fields on  $\text{Hess}_{\mathcal{R}}(H)$ . This situation covers a wide variety of cases (for example all Hessenberg data in a root system of type  $A_\ell$ ) but not all of them. Explicit exceptions are given in the  $C_\ell$  case. One of the main motivations for the present study is the observation that  $\text{Hess}_{\mathcal{R}}(H)$  can be realized locally as a stratified nilpotent group that is *not* always of Iwasawa type. Hence our work is an extension of the theories of multicontact maps developed thus far.

The notion of multicontact structure was introduced in [11] and [12] in the context of the homogeneous spaces  $G/P$ . Roughly speaking, it refers to a collection of special sub-bundles of the tangent bundle with the property that their sections generate the whole tangent space by repeated brackets. The selection of the special directions is not only required to satisfy this Hörmander-type condition, but it is also dictated by the stratification of the tangent space  $T_x$  at each point  $x \in G/P$  in terms of restricted root spaces. If for example  $P$  is minimal, then  $T_x$  can be identified with a nilpotent Iwasawa Lie algebra and therefore it may be viewed as the direct sum of all the root spaces associated to the positive restricted roots. Since a positive root is a sum of simple roots, it is natural to expect that the tangent directions along the simple roots will play a special rôle. Indeed, it is proved in [12] that, at least in rank greater than one,  $G$  acts on  $G/P$  by maps whose differential preserves each sub-bundle corresponding to a simple restricted root, or, at worst, it permutes them amongst themselves. It is thus natural to say that  $g \in G$  induces a *multicontact mapping*. The main result in [12] is that the converse statement is also true: a locally defined  $C^2$  multicontact mapping on  $G/P$  is the restriction of the action of a uniquely determined element  $g \in G$ . Hence the boundaries  $G/P$  are (in most cases) *rigid*.

This type of theorem is one in a long standing history of rigidity results, dating back to Liouville. Around 1850, he proved that any  $C^4$  conformal map between domains in  $\mathbb{R}^3$  is necessarily a composition of translations, dilations and inversions in spheres. This amounts to saying that the group  $O(1, 4)$  acts on the sphere  $S^3$  by conformal transformations (and hence locally on  $\mathbb{R}^3$ , by stereographic projection), and then proving that any conformal map between two domains arises as the restriction of the action of some element of  $O(1, 4)$ . The same result also holds in  $\mathbb{R}^n$  when  $n > 3$  (see, for instance, [24]), and with metric rather than smoothness assumptions (see [18]).

A cornerstone in the extension process of Liouville's result is certainly the paper [23] by A. Korányi and H.M. Reimann, where the Heisenberg group  $\mathbb{H}^n$  substitutes the Euclidean space and the sphere in  $\mathbb{C}^n$  with its Cauchy-Riemann structure substitutes the real sphere. The authors study smooth maps whose differential preserves the contact ("horizontal") plane  $\mathbb{R}^{2n} \subset \mathbb{H}^n$  and is in fact given by a multiple of a unitary map. These maps are called conformal by Korányi and Reimann. Their rigidity theorem states that all conformal maps belong to the group  $SU(1, n)$ .

A second step was taken by P. Pansu [25], who proved that in the quaternionic and octonionic analogues of this set-up Liouville's theorem holds under the sole assumption that the map in question preserves a suitable contact structure of codimension greater than one. Similar phenomena have been studied in more general situations: see, for example, [3], [4], [19], [20].

A remarkable piece of work concerning this circle of ideas is [30], by K. Yamaguchi. His approach is at the infinitesimal level and is based on the theory of  $G$  structures, as developed by N. Tanaka [26]. The crucial step in his analysis uses heavily Kostant's Lie algebra cohomology and classification arguments. It is perhaps fair to say that the latest important contribution in this area is the point of view adopted by Cowling, De Mari, Korányi and Reimann in [11] and [12]. As mentioned earlier, they introduce the notion of multicontact mapping. Their results have a non-trivial overlap with those by Yamaguchi, but are independent of classification, rely on entirely elementary techniques and focus on a very important issue: the main step in proving a rigidity result at the Lie algebra level consists in showing that the appropriate system of differential equations has polynomial solutions.

One may reverse the point of view presented above and argue that a rigidity theorem exhibits  $G$  as a group of geometric transformations (of some natural sort) of a homogeneous space of  $G$ . It is then very natural to ask if rigidity phenomena occur in a wider variety of circumstances, for other Lie groups or, rather, for submanifolds of a rigid manifold. In this thesis we consider the case of a large class of Hessenberg submanifolds of  $G/P$ . Hessenberg manifolds were introduced in the mid 80's by G. Ammar and C. Martin [1], [2] in connection with the study of the QR algorithm as a dynamical system on flag manifolds. Let us briefly recall the definition of the simplest Hessenberg manifolds.

If  $G = \mathrm{SL}(n, \mathbb{R})$  and  $P$  is the minimal parabolic subgroup of  $G$  consisting of the unipotent upper-triangular matrices, then  $G/P$  is identified with the flag manifold  $\mathrm{Flag}(\mathbb{R}^n)$ . The elements of  $\mathrm{Flag}(\mathbb{R}^n)$  are the nested sequences  $S_1 \subset S_2 \subset \cdots \subset S_{n-1}$  of linear subspaces of  $\mathbb{R}^n$ , with  $\dim(S_i) = i$ , and the identification takes place by viewing the first  $i$  columns of the matrix  $g \in G$  as a spanning set for  $S_i$ . Clearly, two matrices will identify the same flag if and only if they differ by right multiplication by an upper-triangular matrix with ones along the main diagonal. This shows that  $\mathrm{Flag}(\mathbb{R}^n) \simeq G/P$ .

Fix now a matrix  $A$  and a positive integer  $p \in \{1, 2, \dots, n-1\}$ . Ammar and Martin say that the flag  $S_1 \subset S_2 \subset \dots \subset S_{n-1}$  is a Hessenberg flag of type  $p$  relative to  $A$  if  $AS_i \subset S_{i+p}$  for all  $i = 1, \dots, n-p-1$ . Thus, a Hessenberg flag of type  $p$  is one for which  $A$  shifts a space of the flag into a larger space (one with  $p$  additional dimensions), within the same flag. The set of all Hessenberg flags of type  $p$  is denoted by  $\text{Hess}_p(A)$  and referred to as a Hessenberg manifold. It was proved in [13] that if  $A$  is diagonal and has distinct non-zero eigenvalues, then  $\text{Hess}_p(A)$  is indeed a smooth manifold.

Notice that the defining condition may be formulated by the single matrix equation  $Ag = gR$ , where  $g \in G$  represents the flag and  $R$  is any  $n \times n$  matrix that has no more than  $p$  non-zero sub-diagonals. Indeed, the first  $i$  columns of  $gR$  are in this case a linear combination of the first  $i+p$  columns of  $g$ . A matrix like  $R$  is known in the Numerical Analysis literature as a Hessenberg matrix of type  $p$ . This clarifies the terminology.

Asking that the identity  $Ag = gR$  is satisfied for some Hessenberg matrix of type  $p$  is equivalent to saying  $g^{-1}Ag \in \mathcal{H}_p$ , where  $\mathcal{H}_p$  is the space of all Hessenberg matrices of type  $p$ , a space that is stable under conjugation by elements in  $P$ . A simple but far-reaching observation is that the defining equation can be written as  $\text{Ad}(g^{-1})A \in \mathcal{H}_p$  and then interpreted in Lie-theoretical terms. This leads to a general notion of Hessenberg manifold. The definition makes sense whenever  $\mathcal{H}_p$  is replaced by a vector space  $\mathcal{H}$  in the Lie algebra of  $G$  that contains the Lie algebra of  $P$  and is stable under  $\text{Ad}(P)$ . In the context of semisimple Lie algebras, these  $P$ -modules  $\mathcal{H}$  can be described in terms of root spaces and turn out to be labeled by those subsets  $\mathcal{R}$  of the set of positive restricted roots that satisfy the so-called Hessenbeg condition: if  $\alpha \in \mathcal{R}$  and  $\beta$  is another positive restricted root such that  $\alpha - \beta$  is again a positive restricted root, then  $\alpha - \beta \in \mathcal{R}$ . Thus, one defines a whole class of manifolds. Each element in the class depends on two choices: the Lie algebra element  $A$  and the combinatorial structure  $\mathcal{R}$ , whence the standard notation  $\text{Hess}_{\mathcal{R}}(A)$ . In particular, one recovers the notion of “type  $p$ ” described above by choosing  $\mathcal{R}$  to be the set of positive roots of height less than or equal to  $p$ .

The Hessenberg manifolds were studied primarily by F. De Mari in a series of papers with different collaborators. The basic topological and geometric features in the complex setting were studied with C. Procesi and M. A. Shayman in [15] and [16], whereas the real Hessenberg manifolds and their close connection with the generalized Toda flow were considered in

collaboration with M. Pedroni [14]. Recently, they have been investigated by J. Tymoczko [27] from the point of view of combinatorial topology.

The first problem addressed in this thesis is an appropriate definition of multicontact structure on the Hessenberg manifold, and it is considered in Chapter 4. Here, as in most of the existing literature, we consider  $\text{Hess}_{\mathcal{R}}(H)$  when  $H$  is a regular element in the Cartan subspace of the Lie algebra of  $G$ . The construction of the special sub-bundles requires a careful local description of the manifold, and this is relatively straightforward once the Bruhat decomposition is taken into play. The point is that by means of the Bruhat decomposition, an open and dense subset of  $G/P$  can be identified with the nilpotent group  $N$  occurring in the Iwasawa decomposition  $G=KAN$ . Then the tangent space at the base point, namely the identity  $e \in N$ , is naturally identified with the Lie algebra  $\mathfrak{n}$  of  $N$  and the exponential coordinates enable to endow  $G/P$  with a (local) structure governed by the restricted roots. By writing down the equations that define  $\text{Hess}_{\mathcal{R}}(H)$ , one realizes that in these coordinates it is the graph of a polynomial mapping. Thus one looks at the independent variables as a natural model. In other words, a coordinate subspace of the Euclidean space  $N$  is selected as a substitute of the Hessenberg manifold, that is a “slice”  $S \subset \text{Hess}_{\mathcal{R}}(H)$ . Inside  $S$ , the coordinates that correspond to simple roots are very well visible, and this calls for the correct identification of the special bundles. In this way the multicontact structure is proved to exist and to satisfy all the reasonable properties that it should satisfy.

Next, we present a case-study, namely we take  $G = \text{SL}(4, \mathbb{R})$  and we take  $P$  to be the minimal parabolic subgroup of unipotent lower-triangular matrices. We consider the structure  $\mathcal{R}$  consisting of all the positive roots except the highest one. Equivalently, we look at the standard Hessenberg manifold with  $p = 2$ . The usual realization of a Cartan subspace  $\mathfrak{a}$  is the space of traceless diagonal matrices, and an element  $H \in \mathfrak{a}$  is regular if and only if it has distinct eigenvalues. The specific choice of  $H$  is irrelevant, but we do make a choice for simplicity. In the attempt of understanding what the multicontact mappings are in this case, it is very natural to follow the method developed in [12], namely to look for the vector fields whose flow consists of multicontact local diffeomorphisms.

The study of multicontact vector fields leads immediately to a system of differential equations (for the components of the vector field) that reveals some of the basic principles that appear in the general case, but also some special features. First of all, the system can be seen as a bunch of

systems (in this case two), each associated to a maximal root in  $\mathcal{R}$ , namely a root  $\mu \in \mathcal{R}$  to which no simple root can be added to give another root in  $\mathcal{R}$ . This is a general feature. Secondly, each of the systems exhibits a hierarchic structure: once the component along the maximal root  $\mu$  is known, then all the components labeled by the roots in the “cone” below  $\mu$  are also known, by suitable differentiation. This is also a general feature. Thirdly, each subsystem is identical to the system whose solutions identify the multicontact vector fields on some other  $\tilde{G}/\tilde{P}$  (here  $\tilde{G} = \mathrm{SL}(3, \mathbb{R})$  and  $\tilde{P}$  is its minimal parabolic subgroup of unipotent lower-triangular matrices). Thus, by [11] each of them can be solved. This is a special feature, and is what we refer to as an instance of several Iwasawa models paired together. Finally, the systems overlap, and the core of the analysis consists in understanding what are the consequences of the overlapping. This is again a general problem.

In the case at hand, that is  $G = \mathrm{SL}(4, \mathbb{R})$ , the analysis can be carried out without difficulties and leads to a very interesting answer. The Lie algebra of multicontact vector fields has dimension 9 and is naturally isomorphic to a quotient Lie algebra. Let us describe it. Denote by  $\Sigma_+$  the set of positive restricted roots and by  $\mathcal{C}$  the complement  $\mathcal{C} = \Sigma_+ \setminus \mathcal{R}$ . Consider the vector space direct sums

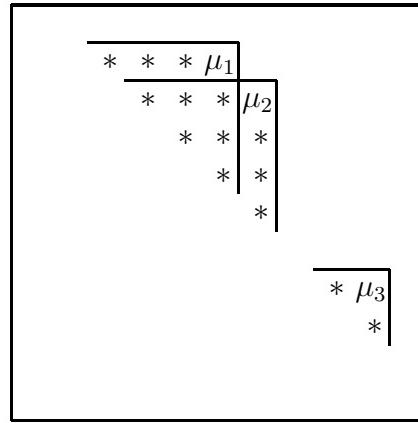
$$\mathfrak{n} = \sum_{\alpha \in \Sigma_+} \mathfrak{g}_\alpha, \quad \mathfrak{n}_{\mathcal{C}} = \sum_{\alpha \in \mathcal{C}} \mathfrak{g}_\alpha,$$

where evidently  $\mathfrak{g}_\alpha$  is the root space associated to  $\alpha$  and  $\mathfrak{n}$  is the usual Iwasawa nilpotent Lie subalgebra of  $\mathfrak{g}$ . Because of the Hessenberg condition,  $\mathfrak{n}_{\mathcal{C}}$  is an ideal in  $\mathfrak{n}$ . Now, let  $\mathfrak{q}$  denote the normalizer of  $\mathfrak{n}_{\mathcal{C}}$  in  $\mathfrak{g}$ , that is the largest subalgebra of  $\mathfrak{g}$  in which  $\mathfrak{n}_{\mathcal{C}}$  is an ideal. In our case,  $\mathcal{C}$  consists of the highest root alone, and the Lie algebra of multicontact vector fields is isomorphic to the quotient  $\mathfrak{q}/\mathfrak{n}_{\mathcal{C}}$ .

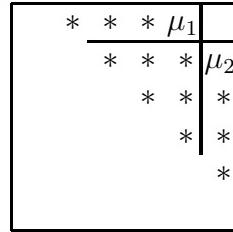
Motivated both by [11], [12] and by the previous example, we define the notion of multicontact vector field on  $\mathrm{Hess}_{\mathcal{R}}(H)$ . Since the nature of our investigations is local, we focus on the local model of  $\mathrm{Hess}_{\mathcal{R}}(H)$ , that is the slice  $S$ . We ask ourselves the following basic question: what is the structure of the Lie algebra  $MC(S)$  of multicontact vector fields on  $S$  in terms of the combinatorial data  $\mathcal{R}$ ? The remaining part of the thesis is devoted to giving a partial answer to this question. We explain below the main steps and keep in mind the case  $G = \mathrm{SL}(n, \mathbb{R})$ .

A simple look at the combinatorics of  $\mathcal{R}$  suggests to partition it in what one is naturally inclined to think of as a connected component. This is

what we shall call “dark zones”. The reason for this terminology is that each of these components may be viewed as the (overlapping) union of shadows, each of which stems from a maximal root  $\mu$ . In the  $\mathfrak{sl}(n, \mathbb{R})$  case, the picture explains the wording.

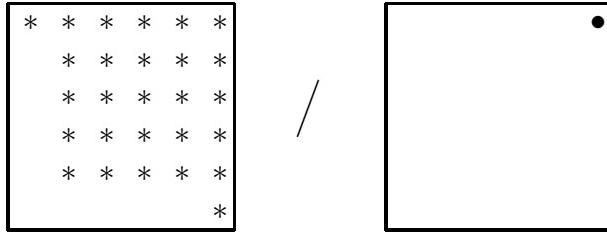


These partitions of course reflect the nature of the differential equations that the components of a multicontact vector field must satisfy. First of all, a vector field has components labeled by  $\mathcal{R}$ ; secondly, the dark zones correspond to independent subsystems and, thirdly, each shadow exhibits a hierachic structure. As a consequence of the mutual independence of dark zones, we may prove a reduction theorem, Theorem 15, that unables us to safely assume that  $\mathcal{R}$  is a single dark zone. This really means that all the simple roots are in  $\mathcal{R}$ . From now on we thus work under this assumption.



At this point, the hierarchy plays a key role. If the components along the maximal roots satisfy the appropriate differential equations, then all the components in the shadow below them are obtained by differentiation. Therefore, the problem reduces to analyzing the differential equations for the maximal roots on one hand, and of understanding the implications of the overlapping shadows on the other hand. By doing so, we obtain a first

result on the Lie algebra  $MC(S)$  of the multicontact vector fields, namely that  $MC(S)$  contains  $\mathfrak{q}/\mathfrak{n}_C$ . This latter Lie algebra corresponds to filling the previous picture with the stars that label the normalizer  $\mathfrak{q}$ , and then taking the quotient modulo the ideal  $\mathfrak{n}_C$ , that is the black dot.



In the general case, however, the converse inclusion  $MC(S) \subseteq \mathfrak{q}/\mathfrak{n}_C$  does not hold, as explained in detail in Section 4.

A natural assumption under which the converse does hold, is that each shadow defines a subalgebra, necessarily an Iwasawa algebra in its own right. One may use Theorem 4.1 in [12] for each single shadow. This is done in the final step, Theorem 22, where we also glue all the different pieces together. This requires a technical description (Lemma 27) of the normalizer  $\mathfrak{q}$  in terms of roots. We finally draw some conclusions concerning the group of multicontact diffeomorphisms. In Proposition 30 we show that it contains canonically the quotient  $Q/N_C$ , where  $Q = \text{Int}(\mathfrak{q})$  and  $N_C = \exp \mathfrak{n}_C$ .

This thesis also contains a chapter devoted to some decomposition results for the polynomials that generate the Lie algebra of multicontact vector fields on  $G/P$ , under the further assumption that  $\mathfrak{g}$  is a split form. We believe that this is of some independent interest and that it indicates another possible area of investigation, namely the explicit description of all the (special) polynomial algebras that the theory of multicontact vector fields seems to produce. For clarity and internal consistency, this part actually precedes the study of Hessenberg manifolds and is developed in Chapter 2. Finally, Chapters 1 and 3 collect some prerequisites.

## CHAPTER 1

### Preliminaries

In this chapter we introduce the fundamental tools that are used in this thesis. In particular, in the first section we recall some very well-known facts about simple Lie algebras, and we fix the notations that will be used throughout. After that, we shall discuss some recent results obtained by Cowling, De Mari, Koranyi and Reimann on the contact structures generalized to the boundaries of symmetric spaces of the type G/P. In the third and last section we illustrate these results in one example.

#### 1. Simple Lie algebras

We shall work with real simple Lie algebras, although most of what we do holds, *mutatis mutandis*, for semisimple Lie algebras. For the reader's convenience, we collect some facts about simple Lie algebras and their decompositions. For more details the reader can look at [5], [21], [28].

Let  $\mathfrak{g}$  be a simple Lie algebra with Cartan involution  $\theta$ . Let  $\mathfrak{k} \oplus \mathfrak{p}$  be the Cartan decomposition of  $\mathfrak{g}$ , where  $\mathfrak{k} = \{X \in \mathfrak{g} : \theta X = X\}$  and  $\mathfrak{p} = \{X \in \mathfrak{g} : \theta X = -X\}$ . Fix a maximal abelian subspace  $\mathfrak{a}$  of  $\mathfrak{p}$ . The dimension of  $\mathfrak{a}$  is an invariant of  $\mathfrak{g}$  and is called the real rank of  $\mathfrak{g}$ . Denote by  $\mathfrak{a}'$  the dual of  $\mathfrak{a}$ . For  $\alpha \in \mathfrak{a}'$ , set

$$(1) \quad \mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [H, X] = \alpha(H)X, \forall H \in \mathfrak{a}\}.$$

When  $\alpha \neq 0$  and  $\mathfrak{g}_\alpha$  is not trivial,  $\alpha$  is said to be a restricted<sup>1</sup> root of  $\mathfrak{g}$ . Denote by  $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$  the set of restricted roots of  $\mathfrak{g}$ . It satisfies all the axioms of a (not necessarily reduced) root system in the usual sense [5]; we refer to it as the root system of  $\mathfrak{g}$ . The basic decomposition of  $\mathfrak{g}$  is then the so-called restricted root space decomposition

$$(2) \quad \mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha,$$

---

<sup>1</sup>The reason of the adjective “restricted” resides in the fact that restricted roots arise as restrictions to  $\mathfrak{a}$  of the roots relative to the Cartan subalgebra  $\mathfrak{h} = (\mathfrak{a} + \mathfrak{t})^c$  of the complexification of  $\mathfrak{g}$ , where  $\mathfrak{t}$  is a maximal abelian subspace of  $\mathfrak{m}$ , as defined in (3).

where  $\mathfrak{m}$  is the centralizer of  $\mathfrak{a}$  in  $\mathfrak{k}$ , that is

$$(3) \quad \mathfrak{m} = \{X \in \mathfrak{k} : [X, H] = 0, H \in \mathfrak{a}\}.$$

Observe that  $\mathfrak{m} \oplus \mathfrak{a} = \mathfrak{g}_0$ , the space that corresponds to the choice  $\alpha = 0$  in (1). Also, we remark that the direct sums in (2) only concern the vector space structure.

Fix a partial ordering  $\succ$  on  $\Sigma$  and denote by  $\Sigma_+$  the subset of  $\mathfrak{a}'$  of positive restricted roots. The space  $\mathfrak{a}'$  is endowed with the inner product  $(\cdot, \cdot)$  induced by the Killing form  $B$  of  $\mathfrak{g}$ . It is defined by  $(\alpha, \beta) = B(H_\alpha, H_\beta)$ , where  $H_\alpha$  is the element of  $\mathfrak{a}$  that represents the functional  $\alpha$  via the Killing form, that is  $\alpha(H) = B(H, H_\alpha)$  for all  $H \in \mathfrak{a}$ . From the Jacobi identity one immediately gets that  $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ , provided that  $\alpha + \beta$  is a root. Choose a basis for each restricted root space. The above version of the Jacobi identity allows us to define the structure constants  $c_{\alpha,\beta}$ , that is the real numbers that satisfy  $[X_\alpha, X_\beta] = c_{\alpha,\beta} X_{\alpha+\beta}$ .

A positive root  $\alpha$  is called simple if it cannot be written as a sum of positive roots. If  $\Delta = \{\delta_1, \dots, \delta_r\}$  denotes the set of simple roots, then the cardinality  $r$  of  $\Delta$  is equal to the real rank of  $\mathfrak{g}$ . Every  $\alpha \in \Sigma_+$  is a linear combination of elements of  $\Delta$  with coefficients in  $\mathbb{N} \cup \{0\}$ . Thus every positive root  $\alpha$  can be written as  $\alpha = \sum_{i=1}^r n_i \delta_i$  for uniquely defined non-negative integers  $n_1, \dots, n_r$ , and the positive integer  $\text{ht}(\alpha) = \sum_{i=1}^r n_i$  is called the height of  $\alpha$ . It is well-known that there is exactly one root  $\omega$ , called the highest root, that satisfies  $\omega \succ \alpha$  (strictly) for every other root  $\alpha$ .

Of central importance in the present context are the nilpotent Lie algebra

$$(4) \quad \mathfrak{n} = \bigoplus_{\gamma \in \Sigma_+} \mathfrak{g}_\gamma$$

and its counterpart  $\bar{\mathfrak{n}} = \theta(\mathfrak{n})$ . For instance  $\mathfrak{n}$  appears in one of the most useful features in the theory of semisimple Lie algebras, the Iwasawa decomposition of  $\mathfrak{g}$ , namely  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ . We shall thus refer to  $\mathfrak{n}$  as the nilpotent Iwasawa subalgebra of  $\mathfrak{g}$ .

One of the main ingredients in the theory of multicontact mappings as developed in [12] is the nature of the index set  $\Sigma_+$  that labels the direct sum (4) or, more importantly, of the corresponding direct summands  $\mathfrak{g}_\alpha$  as  $\alpha$  runs in  $\Sigma_+$ . Indeed, it provides what we call a *multistratification*, as we now briefly explain. The notion of stratified Lie algebra refers properly to

the strata

$$(5) \quad \mathfrak{n}_i = \bigoplus_{\text{ht}(\gamma)=i} \mathfrak{g}_\gamma, \quad i = 1, \dots, \text{ht}(\omega).$$

because they satisfy

$$[\mathfrak{n}_i, \mathfrak{n}_j] \subset \mathfrak{n}_{i+j},$$

as a further application of the Jacobi identity shows. Since the “ground” stratum  $\mathfrak{n}_1$  is the direct sum of the root spaces that are labeled by the simple roots and since each positive restricted root is the sum of simple ones, it follows that  $\mathfrak{n}_1$  generates  $\mathfrak{n}$  as a Lie algebra. Thus  $\mathfrak{n}$  is a stratified Lie algebra in the usual sense (see [17]). What is more important is the fact that  $\mathfrak{n}_1$ , and every higher stratum, is a direct sum of finer building blocks, as indicated in (5). The way in which these finer blocks (i.e. the restricted root spaces) behave under bracket is governed by the root system, which has a highly non-trivial structure. This is the multistratification.

## 2. Multicontact mappings on G/P

In this section we recall some results contained in [12]. In particular, we quote a theorem that plays a fundamental role in this thesis and then we describe the steps that lead to the proof of it.

Let  $\mathfrak{g}$  be as above and let  $G$  be a Lie group whose Lie algebra is  $\mathfrak{g}$ . Let  $P$  be a minimal parabolic subgroup of  $G$ . We may assume that the center of  $G$  is trivial. Indeed, if  $Z$  is the center of  $G$ , then  $Z \subset P$ , and so  $G/P$  and  $(G/Z)/(P/Z)$  may be identified. Moreover, the action of  $G$  on  $G/P$  factors to an action of  $G/Z$ .

Among all groups with trivial centers and the same Lie algebra  $\mathfrak{g}$ , the largest is the group  $\text{Aut}(\mathfrak{g})$  of all automorphisms of  $\mathfrak{g}$ , and the smallest is the group  $\text{Int}(\mathfrak{g})$  of the inner automorphisms of  $\mathfrak{g}$ , the connected component of the identity of  $\text{Aut}(\mathfrak{g})$ . Any group  $G_1$  such that  $\text{Int}(\mathfrak{g}) \subseteq G_1 \subseteq \text{Aut}(\mathfrak{g})$ , with corresponding minimal parabolic subgroup  $P_1$ , gives rise to the same space, meaning that  $G_1/P_1$  may be identified with  $\text{Aut}(\mathfrak{g})/P$  if  $P$  is a minimal parabolic subgroup of  $\text{Aut}(\mathfrak{g})$ . For the purposes of this thesis the correct assumption is that  $G$  is connected and centerless, and hence we can assume  $G = \text{Int}(\mathfrak{g})$  and that  $P$  is a minimal parabolic subgroup of  $G$ .

The most natural choice for  $P$  is as follows. According to the notation introduced in Section 1, let  $\mathfrak{m}$  denote the centralizer of  $\mathfrak{a}$  in  $\mathfrak{k}$  as previously defined. As easily verified,  $\mathfrak{m}$  is the Lie algebra of the centralizer of  $\mathfrak{a}$  in  $K$ , the latter being the connected Lie subgroup of  $G$  whose Lie algebra is

¶. In other words

$$M = \{m \in K : \text{Ad } m(H) = H, H \in \mathfrak{a}\}.$$

Let now  $A = \exp \mathfrak{a}$  and  $N = \exp \mathfrak{n}$  denote the connected (and simply connected) Lie subgroups of  $G$  with Lie algebras  $\mathfrak{a}$  and  $\mathfrak{n}$ , respectively. Finally, put  $\bar{N} = \exp \bar{\mathfrak{n}}$ . Then  $P = M\bar{N}$  is the minimal parabolic subgroup of  $G$  that we shall be concerned with, and the latter expression is the Langlands decomposition of it.

By means of the Bruhat decomposition (see [21], Ch.VII, Sec.4) the group  $N$  may be seen as open and dense in  $G/P$ . Indeed, if we denote by  $b$  the base point in  $G/P$  (that is, the identity coset), the Bruhat lemma states that the mapping  $\psi : N \rightarrow G/P$  defined by  $\psi(n) = nb$  is injective and its image is dense and open. The differential  $\psi_*$  then maps  $\mathfrak{n}$ , the tangent space to  $N$  at the identity  $e$ , onto  $T_b$ , the tangent space to  $G/P$  at the base point. When  $\delta$  is a simple restricted root, we denote by  $S_{\delta,b}$  the subspace  $\psi_*(\mathfrak{g}_\delta)$  of  $T_b$ . In Lemma 2.2 of [12] it is shown that the action of any element  $p \in P$  on  $G/P$  induces an action  $p_*$  on the tangent space  $T_b$  which in turn induces an action  $\psi_*^{-1}p_*\psi_*$  on  $\mathfrak{n}$ . This last action preserves all the spaces  $\mathfrak{g}_\delta$  for simple  $\delta$ . This lemma allows us to identify  $\mathfrak{n}$  with the tangent space  $T_x$  at any point  $x$  in  $G/P$ , and to identify the subspaces  $\mathfrak{g}_\delta$  of  $\mathfrak{n}$  with subspaces  $S_{\delta,x}$  of  $T_x$ . Indeed we may write  $x$  as  $gb$ , where  $g \in G$ ; then the images  $g_*\psi_*\mathfrak{g}_\delta$  are well defined, and independent of the representative  $g$  of the coset, although the identification  $\mathfrak{g}_\delta \rightarrow S_{\delta,x}$  does depend on the representative. Since we never make use of the explicit identification, we shall always write  $\mathfrak{g}_\delta$  in place of  $S_{\delta,x}$ .

Consider a diffeomorphism  $f$  between open subsets of  $G/P$ , say  $\mathcal{U}$  and  $\mathcal{V}$ . By density, we can assume that  $\mathcal{U}$  and  $\mathcal{V}$  are subsets of  $N$ . Then,  $f$  is called a *multicontact* map if  $f_*$  maps  $\mathfrak{g}_\delta$  in itself, for every simple root  $\delta$ . In fact, the original definition given in [12] is slightly weaker, and is designed to cover a wider class of situations, essentially allowing some disconnectedness of  $G$ . The authors of [12] allow  $f_*$  to permute the various  $\mathfrak{g}_\delta$  for simple  $\delta$ . In their context, with  $G = \text{Aut}(\mathfrak{g})$ , they prove that every multicontact mapping on  $G/P$  is the restriction of the action of a uniquely determined element  $g \in G$ . The crucial point of their proof is to focus on the infinitesimal analogue of the notion of multicontact map, where no permutation comes into play. In our setting, this latter notion is even more important, and we actually take it as the basic notion. For this reason we recall it in full detail.

The starting point is to consider multicontact vector fields, that is, vector fields  $V$  on  $\mathcal{U}$  whose local flow  $\{\phi_t^V\}$  consists of multicontact maps. If  $X_\delta \in \mathfrak{g}_\delta$  and  $\delta$  is a simple root, then

$$\frac{d}{dt}(\phi_t^V)_*(X_\delta)\Big|_{t=0} = -\mathcal{L}_V(X_\delta) = [X_\delta, V],$$

where  $\mathcal{L}$  denotes the Lie derivative. Thus a smooth vector field  $V$  on  $\mathcal{U}$  is a multicontact vector field if and only if

$$(6) \quad [V, \mathfrak{g}_\delta] \subseteq \mathfrak{g}_\delta \quad \text{for every simple root } \delta.$$

This is to be interpreted in the sense that, if  $Y \in \mathfrak{g}_\delta$ , then  $[V, Y]$  is a section of the subbundle  $\mathfrak{g}_\delta$  of  $T\mathcal{U}$ , as explained above.

Define a representation  $\tau$  of the Lie algebra  $\mathfrak{g}$  as a set of vector fields on  $N$  as follows:

$$(\tau(X)f)(n) = \frac{d}{dt}f([\exp(tX)n])\Big|_{t=0}.$$

Here  $[\exp(tX)n]$  denotes the action of  $G$  on  $G/P$ , restricted to an action on  $N$ . Hence  $[\exp(tX)n]$  is the  $N$ -component of the product  $\exp(tX) \cdot n$  in the Bruhat decomposition of  $G/P$ . The infinitesimal analogue of the main result contained in [12] is Theorem 4.1, that we recall here.

**THEOREM 1.** ([12]) *Suppose that  $\mathfrak{g}$  has real rank at least two. Then every  $C^1$  multicontact vector field is in fact smooth, and the Lie algebra of multicontact vector fields on  $\mathcal{U}$  consists of the restrictions of  $\tau(\mathfrak{g})$  to  $\mathcal{U}$ .*

The action of  $G$  on  $N$  is multicontact by construction. Hence,  $\tau(\mathfrak{g})$  gives multicontact vector fields by definition. Theorem 1 is thus about the converse implication, which is proved in several steps. Although in the end the basic assertions that lead to the proof are (essentially) coordinate-free, they are best formulated in terms of a chosen canonical basis, as we describe next. For every  $\alpha \in \Sigma_+$ , let  $m_\alpha$  denote the dimension of  $\mathfrak{g}_\alpha$  and fix a basis  $\{X_{\alpha,i} : \alpha \in \Sigma_+, i = 1, \dots, m_\alpha\}$  of  $\mathfrak{n}$  consisting of left-invariant vector fields on  $N$ . Thus, a smooth vector field  $V$  on  $\mathcal{U}$  can be written as

$$(7) \quad V = \sum_{\alpha \in \Sigma_+} \sum_{i=1}^{m_\alpha} v_{\alpha,i} X_{\alpha,i},$$

for some smooth functions  $v_{\alpha,i}$ . The main steps in the proof of Theorem 1 are summarized in the following statements:

- (i) a multicontact vector field is determined by its component in the direction of the highest root, namely  $\{v_{\omega,i} : i = 1, \dots, m_\omega\}$ , and

these functions are determined by a particular set of differential equations;

- (ii) the differential equations imply that any  $v_{\omega,i}$  is a polynomial function in canonical coordinates;
- (iii) the polynomial nature of the functions  $v_{\omega,i}$  implies Theorem 1 by general homogeneity arguments.

The first two steps tell us that any multicontact vector field corresponds to a vector  $(v_{\omega,1}, \dots, v_{\omega,m_\omega})$  of polynomials. Steps (i) and (ii) is where most of the hard work goes. It involves a careful analysis of the system of differential equations via a detailed study of several properties of the restricted root system.

We next discuss in detail the homogeneity concept that is used in the argument of step (iii).

We select an element  $H_0$  in the Cartan subspace  $\mathfrak{a}$  such that  $\delta(H_0) = -1$  for all simple roots  $\delta$  (this is possible because the Cartan matrix is non-singular [21]). A function  $v$  on  $N$  is said to be homogeneous of degree  $r$  if it does not vanish identically and if it also satisfies  $\tau(H_0)v = rv$ . A vector field  $V$  is said to be homogeneous of degree  $s$  if it does not vanish identically and it satisfies  $[\tau(H_0), V] = sV$ . Hence

$$\begin{aligned} \deg(vV) &= \deg(V) + \deg(v), \\ \deg(V(v)) &= \deg(v) + \deg(V) \quad (\text{except when } V(v) = 0), \\ \deg([V, W]) &= \deg(V) + \deg(W) \quad (\text{except when } V \text{ and } W \text{ commute}). \end{aligned}$$

All left invariant vector fields  $X$ , where  $X$  is in the stratum  $\mathfrak{n}_j$ , are of degree  $-j$ . Indeed, write  $n_s = \exp(-sH_0)n \exp(sH_0)$ . Then

$$\begin{aligned} [\tau(H_0), X]f(n) &= \frac{d}{ds} \frac{d}{dt} (f([\exp(-sH_0)n] \exp(tE)) \\ &\quad - f(\exp(-sH_0)[n \exp(tE)])) \Big|_{t=s=0} \\ &= \frac{d}{ds} \frac{d}{dt} (f(\exp(-sH_0)n \exp(sH_0) \exp(tE)) \\ &\quad - f(\exp(-sH_0)n \exp(tE) \exp(sH_0))) \Big|_{t=s=0} \\ &= \frac{d}{ds} \frac{d}{dt} (f(n_s \exp(tE)) - f(n_s \exp(e^{js}tE))) \Big|_{t=s=0} \\ &= \frac{d}{ds} (Xf(n_s) - e^{js}Xf(n_s)) \Big|_{s=0} \\ &= -jXf(n). \end{aligned}$$

If  $X$  lies in a root space  $\mathfrak{g}_\alpha$ , we have

$$[\tau(H_0), \tau(X)] = \tau([H_0, X]) = \alpha(H_0)\tau(X) = -k\tau(X),$$

where  $k$  is the height of  $\alpha$ . Thus, all homogeneous vector fields in  $\tau(\mathfrak{g})$  have degree between  $-h$  and  $h$ , where  $h$  is the height of  $\mathfrak{n}$ , that is, the length of its stratification.

Let us go back to step (iii) in the proof of Theorem 1. First, one observes that a multicontact vector field  $V$  can be written as the sum of its homogeneous parts; together with  $V$ , these also satisfy the differential equations that define the multicontact conditions, i.e., they are multicontact (see Section 3.2 in [12]). Thus,  $V$  can be assumed to be homogeneous. The proof proceeds by treating separately negative, positive and zero degrees. We reproduce below this final argument.

**Negative degree.** Fix  $Y$  in  $\mathfrak{n}_1$  and assume that  $\deg V < 0$ . Then

$$\deg([Y, V]) = \deg(Y) + \deg(V) = -1 + \deg(V) < -1,$$

so  $[Y, V] \in \mathfrak{n}_1$  cannot be a section of the subbundle of  $T\mathcal{U}$  associated to  $\mathfrak{n}_1$  unless  $[Y, V] = 0$ . Therefore  $[Y, V] = 0$  for all  $Y \in \mathfrak{n}_1$ , and since  $\mathfrak{n}_1$  generates  $\mathfrak{n}$ , it follows that  $[Y, V] = 0$  for all  $Y \in \mathfrak{n}$ . If a vector field commutes with infinitesimal right translations, then it is an infinitesimal left translation. Consequently,  $V = \tau(X)$  for some  $X$  in  $\mathfrak{n}$ .

**Negative degree.** Denote by  $M'$  the normalizer in  $K$  of  $\mathfrak{a}$ , namely

$$M' = \{k \in K : \text{Ad } k(H) \in \mathfrak{a}, H \in \mathfrak{a}\},$$

and let  $W$  be the Weyl group  $M'/M$  (see [5]). To treat the case where  $\deg(V) > 0$ , we consider the inversion map  $s$  on  $N$ , induced by the action of  $m'$  on  $G/P$ , where  $m' \in M'$  is a representative of the longest Weyl group element. The induced map  $s_*$  has the property that  $\deg(s_*V) = -\deg(V)$  for homogeneous vector fields  $V$ . Consequently, if  $V$  is a homogeneous multicontact vector field of positive degree, then  $s_*V$  is multicontact (because  $s$  is multicontact) and hence polynomial. Now  $s_*V$  is of negative degree, so  $s_*V = \tau(X)$  for some  $X$  in  $\mathfrak{n}$ , whence  $V = \tau(s_*^{-1}X)$ .

**Zero degree.** Finally, if  $\deg(V) = 0$ , then  $\text{ad } V$  preserves both  $\tau(\mathfrak{n})$  and  $\tau(\bar{\mathfrak{n}})$ , that is,  $\tau(\mathfrak{g})$ . As  $\text{ad } V$  is a derivation of the semisimple Lie algebra  $\tau(\mathfrak{g})$ , there exists  $Y$  in  $\mathfrak{g}$  such that  $V - \tau(Y)$  commutes with  $\tau(\mathfrak{g})$ . If a vector field commutes with  $\tau(\mathfrak{g})$ , then in particular it commutes with infinitesimal left translations, so it is an infinitesimal right translation, and since it also commutes with dilations, it is zero. Hence  $V = \tau(Y)$ . This concludes the proof of Theorem 1.

### 3. A case study: $\mathfrak{sl}(3, \mathbb{R})$

We report an example contained in [11]. We describe this case study in order to illustrate the techniques and the results we presented in the previous section. Moreover, this basic example is crucial for our aims. Indeed, we shall use the results we report here in Chap. 4.

Let  $G = \mathrm{SL}(3, \mathbb{R})$ , and let  $P$  be the minimal parabolic subgroup of  $G$  of lower triangular matrices. For  $x, y$  and  $u$  in  $\mathbb{R}$ , denote by  $\nu(x, y, u)$  the matrix

$$\begin{bmatrix} 0 & x & u \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix}.$$

Take  $\alpha$  and  $\beta$  to be the simple roots relative to the standard Cartan subalgebra of  $\mathfrak{sl}(3, \mathbb{R})$  of diagonal matrices:  $\alpha(\mathrm{diag}(a, b, c)) = (a - b)$  and  $\beta((\mathrm{diag}(a, b, c))) = (b - c)$ . Then

$$\begin{aligned} \mathfrak{g}_\alpha &= \{\nu(x, 0, 0) : x \in \mathbb{R}\}, \\ \mathfrak{g}_\beta &= \{\nu(0, y, 0) : y \in \mathbb{R}\}, \\ \mathfrak{g}_{\alpha+\beta} &= \{\nu(0, 0, u) : u \in \mathbb{R}\}. \end{aligned}$$

Further,  $\mathfrak{n} = \{\nu(x, y, u) : x, y, u \in \mathbb{R}\}$ ; the algebra  $\mathfrak{n}$  has the multistratification  $\mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta \oplus \mathfrak{g}_{\alpha+\beta}$  and the stratification  $\mathfrak{n}_1 \oplus \mathfrak{n}_2$ , where  $\mathfrak{n}_1 = \mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta$  and  $\mathfrak{n}_2 = \mathfrak{g}_{\alpha+\beta}$ .

We write a vector field  $V$  on  $\mathcal{U}$  (open subset of  $N$ ) as  $fX + gY + hU$ , where  $f, g$  and  $h$  are smooth functions on  $\mathcal{U}$  in the coordinates  $x, y, u$  and where  $\{X, Y, U\}$  is the canonical basis of  $\mathfrak{n}$ . Viewed as left-invariant vector fields, they are

$$X = \frac{\partial}{\partial x}, \quad Y = \frac{\partial}{\partial y} + x \frac{\partial}{\partial u}, \quad U = \frac{\partial}{\partial u}.$$

Clearly

$$[X, Y] = U$$

is the only non-zero bracket. We ask ourselves when  $V$  is a multicontact vector field. The multicontact vector field equations (6) state that this happens if and only if  $[V, X] = \lambda X$  and  $[V, Y] = \mu Y$ , for some smooth functions  $\lambda$  and  $\mu$  on  $\mathcal{U}$ . These equations imply immediately that

$$\begin{aligned} \lambda X &= -gU - (Xf)X - (Xg)Y - (Xh)U \\ \mu Y &= fU - (Yf)X - (Yg)Y - (Yh)U, \end{aligned}$$

which in turn imply that

$$\begin{aligned} Xf &= -\lambda & Yg &= -\mu \\ Xg &= 0 & Yf &= 0 \\ Xh &= -g & Yh &= f. \end{aligned}$$

We see at once that  $f$  and  $g$  are determined by  $h$  and that  $h$  itself satisfies the differential equations

$$(8) \quad X^2h = Y^2h = 0.$$

The equation  $X^2h = \partial^2h/\partial x^2 = 0$  has the general solution

$$h(x, y, u) = h_0(y, u) + xh_1(y, u),$$

for some functions  $h_0$  and  $h_1$ . The equation  $Y^2h = 0$  then becomes

$$\begin{aligned} 0 &= \left( \frac{\partial^2}{\partial y^2} + 2x \frac{\partial^2}{\partial y \partial u} + x^2 \frac{\partial^2}{\partial u^2} \right) (h_0 + xh_1) \\ &= x^3 \left( \frac{\partial^2 h_1}{\partial u^2} \right) + x^2 \left( \frac{\partial^2 h_0}{\partial u^2} + 2 \frac{\partial^2 h_1}{\partial y \partial u} \right) + x \left( \frac{\partial^2 h_1}{\partial y^2} + 2 \frac{\partial^2 h_0}{\partial y \partial u} \right) + \left( \frac{\partial^2 h_0}{\partial y^2} \right). \end{aligned}$$

Since the right hand side vanishes identically in some open set, the coefficients of the various powers of  $x$  must vanish. Considering the  $x^3$  term, we see that  $\partial^2 h_1 / \partial u^2 = 0$ . Differentiating the coefficient of the  $x^2$  term once with respect to  $u$ , we deduce that  $\partial^3 h_0 / \partial u^3 = 0$ . Next, considering the constant term yields  $\partial^2 h_0 / \partial y^2 = 0$ , and then differentiating the coefficient of the  $x$  term once with respect to  $y$ , we deduce that  $\partial^3 h_1 / \partial y^3 = 0$ . Summarizing, we have shown that

$$\frac{\partial^3}{\partial u^3} h_0 = 0, \quad \frac{\partial^2}{\partial u^2} h_1 = 0, \quad \frac{\partial^2}{\partial y^2} h_0 = 0, \quad \frac{\partial^3}{\partial y^3} h_1 = 0.$$

The first two equations imply that

$$h_0(y, u) = u^2 a(y) + u b(y) + c(y), \quad h_1(y, u) = u d(y) + e(y),$$

and the second two equations then imply that  $a'' = b'' = c'' = d''' = e''' = 0$ , so that both  $h_0$  and  $h_1$  are polynomials, whence  $h$  is too. The calculations we did so far correspond to the steps (i) and (ii) of the proof of Theorem 1, and the techniques here presented can be generalized. Nevertheless, in this case the differential system can be integrated explicitly, and we obtain:

$$(9) \quad \begin{aligned} h(x, y, u) &= c_0 + c_1 x + c_2 y + c_3 u + c_4 x y \\ &\quad + c_5 x(u - x y) + c_6 u y + c_7 u(u - x y), \end{aligned}$$

where  $c_0, \dots, c_7 \in \mathbb{R}$ . At this point, one can easily calculate a basis of  $\tau(\mathfrak{g})$  given by homogeneous vector fields, and check the following correspondence (up to constants) between polynomials and Lie algebra generators, in the sense that the polynomial  $p$  corresponds to the unique multicontact vector field whose  $U$  component is  $pU$ :

$$\begin{array}{ll} U \leftrightarrow 1 & \theta U \leftrightarrow u(u - xy) \\ Y \leftrightarrow x & \theta Y \leftrightarrow yu \\ X \leftrightarrow y & \theta X \leftrightarrow x(u - xy) \\ H_\alpha \leftrightarrow u - 2xy \\ H_\beta \leftrightarrow u + xy, \end{array}$$

where  $H_\alpha$  and  $H_\beta$  are the elements in  $\mathfrak{a}$  that represent the simple roots by the Killing form.

We remark that the polynomials that appear in the second column are products of those in the first column. Indeed, there exists  $H \in \mathfrak{a}$  such that the polynomial corresponding to  $\tau(H)$  is  $u - xy$  (namely  $H = (2/3)H_\alpha + (1/2)H_\beta$ ), and similarly there exists  $H \in \mathfrak{a}$  such that the polynomial corresponding to  $\tau(H)$  is  $u$  (namely  $H = (1/3)H_\alpha + (2/3)H_\beta$ ). We can interpret this property in a general setting, by showing that the polynomials that correspond to the root spaces labeled by the roots in a certain subset generate all the others. This remark motivates the next chapter, where we give explicit factorization formulas for these polynomials in the case that  $\mathfrak{g}$  is a split simple Lie algebra.

## CHAPTER 2

### **Polynomial basis for split simple Lie algebras**

We investigate the polynomial nature of the multicontact vector fields on some open subset of an Iwasawa nilpotent Lie group  $N$ . We consider the vector space  $\mathcal{P}$  consisting of the polynomials that characterize the multicontact vector fields on  $N$ , endowed with the Lie algebra structure induced by the vector space isomorphism of  $\mathcal{P}$  with  $\mathfrak{g}$  [12]. In particular, in the second section we compute some explicit formulas for a basis of  $\mathcal{P}$ , pointing out some factorization properties.

#### **1. The polynomial algebra $\mathcal{P}$**

Let us summarize parts of the discussion of the previous chapter. Theorem 1 asserts that, under the assumption that the real rank of  $\mathfrak{g}$  is greater than one, the Lie algebra of multicontact vector fields is  $\tau(\mathfrak{g})$ . This latter algebra may be viewed as a Lie algebra of polynomials, because in the basis (7), the components along the highest restricted root  $\omega$  are polynomials and determine all the other components. Also, there is a natural notion of homogeneity that comes into the picture, which is very useful in order to describe the polynomials.

Now, let us consider again a basis adapted to the restricted root space decomposition (2). Clearly the Lie algebra of multicontact vector fields is generated by the set

$$\{\tau(X_{\alpha,i}), \alpha \in \Sigma \cup \{0\}, i = 1, \dots, m_\alpha\}.$$

As we just remarked, to any such vector field one associates a vector of polynomials, namely the coefficients along the  $\omega$ -components. We shall consider this problem in the next section.

#### **2. The split case**

Let  $\mathfrak{g}$  be a real split simple Lie algebra, that is, a split real form of its complexification. This means that if  $\mathfrak{g}^c = \mathfrak{g} \oplus i\mathfrak{g}$  is the complexification of  $\mathfrak{g}$ , then there exists a Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}^c$  such that if  $\Phi$  is the set of

roots relative to the pair  $(\mathfrak{g}^c, \mathfrak{h})$ , then  $\mathfrak{g}$  contains the real subspace of  $\mathfrak{h}$  on which all the roots in  $\Phi$  are real, namely

$$\{H \in \mathfrak{h} : \alpha(H) \in \mathbb{R}, \alpha \in \Phi\} \subset \mathfrak{g}.$$

It is well-known that any complex semisimple Cartan subalgebra contains a split real form (see e.g. Corollary 6.10 in [21]). The most relevant consequences of this assumption for our considerations are that  $\mathfrak{m} = \{0\}$  and that each restricted root space has real dimension one. In particular, this implies that  $I(\mathfrak{g}_\alpha)$  consists of the real multiples of a single polynomial.

Our decomposition formulas are relative to a suitable decomposition of the restricted root system that first appeared in [9]. Given two roots  $\alpha$  and  $\beta$ , the  $\alpha$ -series of  $\beta$  is the set  $\{\gamma \in \Sigma \cup \{0\} : \gamma = \beta + n\alpha\}$ . It turns out that the  $\alpha$ -series is an uninterrupted string, namely that  $n$  takes all the integer values in the interval  $[p, q]$ , where the two integers  $p \leq 0$  and  $q \geq 0$  satisfy the equality  $p + q = -2(\beta, \alpha)/(\alpha, \alpha)$ . In particular, for the  $\omega$ -series of any root  $\beta \in \Sigma_+$ , we have  $p \in \{-1, 0\}$ . This means that either  $\omega - \beta \in \Sigma_+$  or  $\omega - \beta$  is not a root, according as  $(\omega, \beta) = \frac{1}{2}(\omega, \omega)$  or  $(\omega, \beta) = 0$ . This gives us the natural decomposition of  $\Sigma_+$  into the disjoint union

$$\Sigma_+ = \Sigma_{1/2} \cup \Sigma_0 \cup \Sigma_1,$$

where

$$\begin{aligned} \Sigma_0 &= \{\beta \in \Sigma_+ : (\omega, \beta) = 0\}, \\ \Sigma_{1/2} &= \{\beta \in \Sigma_+ : (\omega, \beta) = \frac{1}{2}(\omega, \omega)\}, \\ \Sigma_1 &= \{\omega\}. \end{aligned}$$

We shall write  $\Delta_{1/2} = \Sigma_{1/2} \cap \Delta$  and  $\Delta_0 = \Sigma_0 \cap \Delta$ . According to the decomposition of  $\Sigma_+$ , we put

$$\mathfrak{n} = \mathfrak{n}_{(0)} \oplus \mathfrak{n}_{(1/2)} \oplus \mathfrak{n}_{(1)},$$

with obvious notations. Since  $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ , and  $(\alpha + \beta, \omega) = (\alpha, \omega) + (\beta, \omega)$ , one has that  $\mathfrak{n}_{(0)}$  is a subalgebra and  $\mathfrak{n}_{(1/2)} \oplus \mathfrak{n}_{(1)}$  is an ideal in  $\mathfrak{n}$ . Finally, we recall that the Cartan involution  $\theta$  maps each root space  $\mathfrak{g}_\alpha$  to  $\mathfrak{g}_{-\alpha}$ , so that  $\bar{\mathfrak{n}} = \theta\mathfrak{n} = \bigoplus_{\gamma \in \Sigma_-} \mathfrak{g}_\gamma$ , where  $\Sigma_- = -\Sigma_+$ . According to the notations introduced above, we write

$$\bar{\mathfrak{n}} = \bar{\mathfrak{n}}_{(0)} \oplus \bar{\mathfrak{n}}_{(1/2)} \oplus \bar{\mathfrak{n}}_{(1)},$$

so that

$$(10) \quad \mathfrak{g} = \mathfrak{n}_{(1)} \oplus \mathfrak{n}_{(1/2)} \oplus \mathfrak{n}_{(0)} \oplus \mathfrak{a} \oplus \bar{\mathfrak{n}}_{(0)} \oplus \bar{\mathfrak{n}}_{(1/2)} \oplus \bar{\mathfrak{n}}_{(1)}.$$

By the linearity of scalar product, it is easy to check the following commutation rules

$$\begin{aligned}
(11) \quad & [\mathfrak{n}_{(r)}, \mathfrak{a}] \subset \mathfrak{n}_{(r)}, \quad r = 0, \frac{1}{2}, 1 \\
& [\mathfrak{n}_{(0)}, \bar{\mathfrak{n}}_{(0)}] \subset \mathfrak{n}_{(0)} \oplus \mathfrak{a} \oplus \bar{\mathfrak{n}}_{(0)} \\
& [\mathfrak{n}_{(1/2)}, \bar{\mathfrak{n}}_{(0)}] \subset \mathfrak{n}_{(1/2)} \\
& [\mathfrak{n}_{(1)}, \bar{\mathfrak{n}}_{(0)}] = \{0\} \\
& [\mathfrak{n}_{(0)}, \bar{\mathfrak{n}}_{(1/2)}] \subset \bar{\mathfrak{n}}_{(1/2)} \\
& [\mathfrak{n}_{(1/2)}, \bar{\mathfrak{n}}_{(1/2)}] \subset \mathfrak{n}_{(0)} \oplus \mathfrak{a} \oplus \bar{\mathfrak{n}}_{(0)} \\
& [\mathfrak{n}_{(1)}, \bar{\mathfrak{n}}_{(1/2)}] \subset \mathfrak{n}_{(1/2)} \\
& [\mathfrak{n}_{(0)}, \bar{\mathfrak{n}}_{(1)}] = \{0\} \\
& [\mathfrak{n}_{(1/2)}, \bar{\mathfrak{n}}_{(1)}] \subset \bar{\mathfrak{n}}_{(1/2)} \\
& [\mathfrak{n}_{(1)}, \bar{\mathfrak{n}}_{(1)}] \subset \mathfrak{a}.
\end{aligned}$$

The proof of the next lemma is based on the above rules, and leads us to a key explicit formula. Take the following canonical coordinates on  $N$ :

$$n = n_1 n_{1/2} n_0 = \exp(zZ) \exp\left(\sum_{\alpha \in \Sigma_{1/2}} y_\alpha Y_\alpha\right) \exp\left(\sum_{\beta \in \Sigma_0} x_\beta X_\beta\right),$$

where  $\{X_\beta : \beta \in \Sigma_0\}$  and  $\{Y_\alpha : \alpha \in \Sigma_{1/2}\}$  are a basis of  $\mathfrak{n}_{(0)}$  and  $\mathfrak{n}_{(1/2)}$  respectively, and  $Z \in \mathfrak{n}_{(1)}$ .

**LEMMA 2.** *Let  $X \in \mathfrak{g}_\alpha$ ,  $\alpha \in \Sigma \cup \{0\}$ , and  $n$  in  $N$ . By the Bruhat lemma, for  $t$  small enough there exists  $b(t) \in P$  such that  $\exp(tX)nb(t) \in N$ . Consider the decomposition of  $n^{-1} \exp(tX)nb(t)$  with respect to the chosen coordinates, namely*

$$n^{-1} \exp(tX)nb(t) = n_1^X(t) n_{1/2}^X(t) n_0^X(t).$$

Write  $n = n_1 n_{1/2} n_0$ , then

(i) *there exists  $A \in \mathfrak{n}_{(1)}$  and  $B \in \mathfrak{n}_{(1/2)} \oplus \mathfrak{n}_{(0)} \oplus \mathfrak{a} \oplus \bar{\mathfrak{n}}$  such that*

$$n_{1/2}^{-1} n_1^{-1} \exp(tX) n_1 n_{1/2} = \exp(tA) \exp(tB) \exp(o(t));$$

(ii)

$$\frac{d}{dt} (n_1^X(t)) \Big|_{t=0} = A.$$

**PROOF.** Write

$$n^{-1} \exp(tX)n = n_0^{-1} n_{1/2}^{-1} n_1^{-1} \exp(tX) n_1 n_{1/2} n_0.$$

Observe first that since  $n_1 = \exp(zZ)$ ,

$$n_1^{-1} \exp(tX)n_1 = \exp(e^{-\text{ad}(zZ)}tX).$$

Now, by (11)

$$[Z, X] \in \begin{cases} \mathfrak{n}_{(1)} & \text{if } X \in \mathfrak{a} \\ \mathfrak{n}_{(1/2)} & \text{if } X \in \overline{\mathfrak{n}}_{(1/2)} \\ \mathfrak{a} & \text{if } X \in \overline{\mathfrak{n}}_{(1)}, \end{cases}$$

and if  $X$  belongs to some other summand in the decomposition (10), then  $[Z, X] = 0$ . Therefore

$$\begin{aligned} (12) \quad n_1^{-1} \exp(tX)n_1 &= \exp(tX + t(H_1 + A_{1/2} + A_1) + o(t)) \\ &= \exp(tA_1 + o(t)) \exp(tX + t(H_1 + A_{1/2}) + o(t)), \end{aligned}$$

where  $H_1 \in \mathfrak{a}$ ,  $A_{1/2} \in \mathfrak{n}_{(1/2)}$  and  $A_1 \in \mathfrak{n}_{(1)}$ .

Secondly, since  $\mathfrak{n}_{(1)}$  commutes with  $\mathfrak{n}$ , we consider

$$n_{1/2}^{-1} \exp(tX + t(H_1 + A_{1/2}))n_{1/2} = \exp(e^{-\sum_\alpha y_\alpha \text{ad}Y_\alpha}(tX + tH_1 + tA_{1/2})).$$

Recall that  $n_{1/2}$  is obtained exponentiating some element in  $\mathfrak{n}_{(1/2)}$ . Thus, in the above formula  $\alpha \in \Sigma_{1/2}$ , so that if the commutator  $[Y_\alpha, X] \neq 0$ , then by (11) the following possibilities arise:

$$[Y_\alpha, X] \in \begin{cases} \overline{\mathfrak{n}}_{1/2} & \text{if } X \in \overline{\mathfrak{n}}_{(1)} \\ \mathfrak{a} & \text{if } X \in \overline{\mathfrak{n}}_{(1/2)} \\ \mathfrak{n}_{(1/2)} & \text{if } X \in \overline{\mathfrak{n}}_{(0)} \oplus \mathfrak{n}_{(0)} \oplus \mathfrak{a} \\ \mathfrak{n}_{(1)} & \text{if } X \in \mathfrak{n}_{(1/2)}. \end{cases}$$

Moreover,

$$[Y_\alpha, H_1] \in \mathfrak{a}, \quad [Y_\alpha, A_{1/2}] \in \mathfrak{n}_{(1)}.$$

Therefore

$$\begin{aligned} (13) \quad n_{1/2}^{-1} \exp(tX + t(H_1 + A_{1/2}))n_{1/2} &= \exp(tX + t(B_{1/2}^- + H_2 + B_{1/2} + B_1) + o(t)), \\ &= \exp(tB_1 + o(t)) \exp(tX + t(B_{1/2}^- + H_2 + B_{1/2}) + o(t)), \end{aligned}$$

for some  $B_{1/2}^- \in \overline{\mathfrak{n}}_{(1/2)}$ ,  $H_2 \in \mathfrak{a}$ ,  $B_{1/2} \in \mathfrak{n}_{(1/2)}$  and  $B_1 \in \mathfrak{n}_{(1)}$ . Also, observe that by the Baker-Campbell-Hausdorff formula

$$(14) \quad \exp(tL + o(t)) = \exp(tL) \exp(o(t))$$

for any  $L \in \mathfrak{g}$ . Thus, by (12) and (13) we deduce that

$$n_{1/2}^{-1} n_1^{-1} \exp(tX)n_1 n_{1/2} = \exp(tA) \exp(tB) \exp(o(t)),$$

with

$$A = \begin{cases} A_1 + B_1 & \text{if } X \notin \mathfrak{n}_{(1)} \\ A_1 + B_1 + X & \text{if } X \in \mathfrak{n}_{(1)} \end{cases}$$

and

$$B = \begin{cases} B_{1/2}^- + H_2 + B_{1/2} & \text{if } X \in \mathfrak{n}_{(1)} \\ B_{1/2}^- + H_2 + B_{1/2} + X & \text{if } X \notin \mathfrak{n}_{(1)}. \end{cases}$$

This proves (i). Next, consider

$$n_0^{-1} \exp(tX + t(B_{1/2}^- + H_2 + B_{1/2})) n_0 = \exp(e^{-\text{ad}(\sum_\beta x_\beta X_\beta)}(tX + tB_{1/2}^- + tH_2 + tB_{1/2})).$$

If  $[X_\beta, X] \neq 0$ , then by (11)

$$[X_\beta, X] \in \begin{cases} \overline{\mathfrak{n}}_{(1/2)} & \text{if } X \in \overline{\mathfrak{n}}_{(1/2)} \\ \mathfrak{n}_{(0)} \oplus \overline{\mathfrak{n}}_{(0)} \oplus \mathfrak{a} & \text{if } X \in \mathfrak{n}_{(0)} \oplus \overline{\mathfrak{n}}_{(0)} \oplus \mathfrak{a} \\ \mathfrak{n}_{(1/2)} & \text{if } X \in \mathfrak{n}_{(1/2)}. \end{cases}$$

Furthermore,

$$[X_\beta, B_{1/2}^-] \in \overline{\mathfrak{n}}_{(1/2)}, \quad [X_\beta, H_2] \in \mathfrak{n}_{(0)}, \quad [X_\beta, B_{1/2}] \in \mathfrak{n}_{(1/2)}.$$

Hence

$$(15) \quad \begin{aligned} n_0^{-1} \exp(tX + t(B_{1/2}^- + H_2 + B_{1/2})) n_0 \\ = \exp(tX + t(C_{1/2}^- + C_0^- + H_3 + C_0^+ + C_{1/2}) + o(t)), \end{aligned}$$

for some  $C_{1/2}^- \in \overline{\mathfrak{n}}_{(1/2)}$ ,  $C_0^- \in \overline{\mathfrak{n}}_{(0)}$ ,  $H_3 \in \mathfrak{a}$ ,  $C_0^+ \in \mathfrak{n}_{(0)}$  and  $C_{1/2} \in \mathfrak{n}_{(1/2)}$ .

Recall that  $\mathfrak{n}_{(1)}$  commutes with all  $\mathfrak{n}$ . Thus using (12), (13), (14) and (15), we obtain

$$\begin{aligned} n^{-1} \exp(tX) n &= \exp(tA_1 + tB_1 + o(t)) \times \\ &\quad \times \exp(tX + t(C_{1/2}^- + C_0^- + H_3 + C_0^+ + C_{1/2}) + o(t)) \\ &= \exp(tA_1 + tB_1 + o(t)) \exp(tX + tC_0^+ tC_{1/2} + o(t)) \times \\ &\quad \times \exp(t(C_{1/2}^- + C_0^- + H_3) + o(t)) \\ &= \exp(tA_1 + tB_1 + tk_1(X)) \exp(tC_{1/2} + tk_{1/2}(X)) \times \\ &\quad \times \exp(tC_0 + tk_0(X)) \exp(tC_{1/2}^- + tC_0^- + tH_3 + tk(X)) \times \\ &\quad \times \exp(o(t)) \\ (16) \quad &= \exp(tA) \exp(tC) \exp(tD) \exp(tE) \exp(o(t)), \end{aligned}$$

where

$$k(X) = \begin{cases} X & \text{if } X \in \mathfrak{a} \oplus \overline{\mathfrak{n}} \\ 0 & \text{otherwise ,} \end{cases}$$

$$k_i(X) = \begin{cases} X & \text{if } X \in \mathfrak{n}_{(i)}, i = 0, 1/2, 1 \\ 0 & \text{otherwise ,} \end{cases}$$

and

$$C = C_{1/2} + k_{1/2}(X), \quad D = C_0 + k_0(X), \quad E = C_{1/2}^- + C_0^- + H_3 + k(X).$$

On the other hand, by hypothesis

$$(17) \quad n^{-1} \exp(tX)n = n_1^X(t)n_{1/2}^X(t)n_0^X(t)b(t)^{-1}.$$

Observe that since  $n^{-1} \exp(tX)n$  is the identity for  $t = 0$ , then necessarily  $n_r^X(0) = e$  for every  $r = 1, 1/2, 0$ , and  $b(0) = e$ . Therefore, comparing (16) and (17),

$$\begin{aligned} \frac{d}{dt} (\exp(tA) \exp(tC) \exp(tD) \exp(tE) \exp(o(t))) \Big|_{t=0} \\ = \frac{d}{dt} (n_1^X(t)n_{1/2}^X(t)n_0^X(t)b(t)^{-1}) \Big|_{t=0}, \end{aligned}$$

whence

$$\begin{aligned} A + C + D + E &= \frac{d}{dt} (n_1^X(t)) \Big|_{t=0} n_{1/2}^X(0)n_0^X(0)b(0)^{-1} \\ &\quad + n_1^X(0) \frac{d}{dt} (n_{1/2}^X(t)) \Big|_{t=0} n_0^X(0)b(0)^{-1} \\ &\quad + n_1^X(0)n_{1/2}^X(0) \frac{d}{dt} (n_0^X(t)) \Big|_{t=0} b(0)^{-1} \\ &\quad + n_1^X(0)n_{1/2}^X(0)n_0^X(0) \frac{d}{dt} (b(t)^{-1}) \Big|_{t=0}. \end{aligned}$$

This implies

$$A = \frac{d}{dt} (n_1^X(t)) \Big|_{t=0},$$

because  $A$  and  $\frac{d}{dt} (n_1^X(t)) \Big|_{t=0}$  are the only two terms in the above sum that lie along  $Z$ . Thus also (ii) is proved.  $\square$

Let  $X \in \mathfrak{g}_\alpha$ ,  $\alpha \in \Sigma \cup \{0\}$ . The multicontact vector field associated to  $X$  is defined by

$$\tau(X)f(n) = \frac{d}{dt} f([\exp(tX)n]) \Big|_{t=0},$$

where  $[\exp(tX)n]$  is the  $N$ - component of  $\exp(tX)n$  in the Bruhat decomposition. This is equivalent to saying that for  $t$  small enough there exists  $b(t) \in P$  such that  $[\exp(tX)n] = \exp(tX)nb(t) \in N$ . Hence

$$\begin{aligned}\tau(X)f(n) &= \frac{d}{dt}f(\exp(tX)nb(t))\Big|_{t=0} \\ &= \frac{d}{dt}f(nn^{-1}\exp(tX)nb(t))\Big|_{t=0} \\ &= \frac{d}{dt}f(n_1n_{1/2}n_0n_1^X(t)n_{1/2}^X(t)n_0^X(t))\Big|_{t=0}.\end{aligned}$$

In the last part of the proof of Lemma 2, we observed that  $n_r^X(0) = e$ , for every  $r = 0, 1/2, 1$ . Recalling that  $p$  is the coefficient of  $Z$  in the decomposition of  $\tau(X)$ , by the latter assertion we have

$$(18) \quad p(n) = \frac{d}{dt}(n_1^X(t))\Big|_{t=0}.$$

In particular, by (ii) of Lemma 2 we desume that the calculation of  $p(n)$  consists in computing the conjugation  $n_{1/2}^{-1}n_1^{-1}\exp(tX)n_1n_{1/2}$  and writing it in the form  $\exp(tA)\exp(tB + o(t))$ , with  $A \in \mathfrak{n}_{(1)}$  and  $B \in \mathfrak{g} \setminus \mathfrak{n}_{(1)}$ . In short,  $p(n) = A$ .

We shall obtain explicit formulas for the homogeneous polynomials corresponding to  $\mathfrak{g}$  using (18). We consider separately the cases with  $\alpha$  that lies respectevely in  $\Sigma_0, \Sigma_{1/2}, \Sigma_1, \{0\}, -\Sigma_1, -\Sigma_{1/2}, -\Sigma_0$ . The formulas will point out that the polynomials corresponding to  $\Sigma_0, -\Sigma_{1/2}, -\Sigma_0, -\Sigma_1$  arise as products and suitable linear combinations of those corresponding to the roots in  $\Sigma_{1/2}$  and  $\{0\}$ . Before collecting the formulas in a list of propositions, we still introduce a couple of notations.

We define on the set  $\Sigma_{1/2}$  the equivalence relation  $\sim$  given by

$$\alpha \sim \beta \Leftrightarrow \alpha + \beta = \omega,$$

and we choose one representative for each element of the quotient  $(\Sigma_{1/2}/\sim)$ . Denote the set of such representatives by  $\tilde{\Sigma}_{1/2}$ . From now until the end of this chapter, we write  $p^\alpha$  and  $p^H$  for the polynomial that corresponds to  $X_\alpha \in \mathfrak{g}_\alpha$  and to  $H \in \mathfrak{a}$ , respectevely.

**PROPOSITION 3.** (i) *If  $\gamma \in \Sigma_{1/2}$ , then  $p^\gamma(n) = c_{\gamma,\omega-\gamma}y_{\omega-\gamma}$ .*

(ii) *If  $H \in \mathfrak{a}$ , then*

$$p^H(n) = \omega(H)z - \frac{1}{2} \sum_{[\alpha] \in \Sigma_{1/2}/\sim} y_\alpha y_{\omega-\alpha} ((\omega - \alpha)(H) - \alpha(H)) c_{\alpha,\omega-\alpha}.$$

(iii)  $p^\omega(n) = 1$ .

PROOF. (i) If  $\gamma \in \Sigma_{1/2}$  and  $\alpha$  is another root in  $\Sigma_{1/2}$ , then  $\gamma + \alpha = \omega$  provided  $\gamma + \alpha$  is a root. This implies that  $\alpha = \omega - \gamma$ . Furthermore,  $[Z, \mathfrak{n}] = 0$ . Therefore

$$\begin{aligned} n_{1/2}^{-1} n_1^{-1} \exp(tY_\gamma) n_1 n_{1/2} &= n_{1/2}^{-1} \exp\left(\sum_{n=0}^{+\infty} (-1)^n \frac{(\text{ad } zZ)^n}{n!} tY_\gamma\right) n_{1/2} \\ &= n_{1/2}^{-1} \exp(tY_\gamma) n_{1/2} \\ &= \exp\left(\sum_{n=0}^{+\infty} (-1)^n \frac{(\text{ad}(\sum_{\alpha \in \Sigma_{1/2}} y_\alpha Y_\alpha))^n}{n!} tY_\gamma\right) \\ &= \exp(tY_\gamma - ty_{\omega-\gamma}[Y_{\omega-\gamma}, Y_\gamma]) \\ &= \exp(tc_{\gamma, \omega-\gamma} y_{\omega-\gamma} Z) \exp(tY_\gamma). \end{aligned}$$

By (18) and the remark thereafter, we have

$$p^\gamma(n) = c_{\gamma, \omega-\gamma} y_{\omega-\gamma}.$$

(ii) Since  $[\mathfrak{n}_{(1/2)}, \mathfrak{n}_{(1/2)}] \subseteq \mathfrak{n}_{(1)}$ , every bracket involving three or more vectors in  $\mathfrak{n}_{(1/2)}$  is zero. Then, for  $H \in \mathfrak{a}$ , we have

$$\begin{aligned} n_{1/2}^{-1} n_1^{-1} \exp(tH) n_1 n_{1/2} &= n_{1/2}^{-1} \exp(tH - tz[Z, H]) n_{1/2} \\ &= \exp(t\omega(H)zZ) \times \\ &\quad \times \exp\left(tH - t \sum_{\alpha \in \Sigma_{1/2}} y_\alpha [Y_\alpha, H] + t/2 \sum_{\alpha+\beta=\omega} y_\alpha y_\beta [Y_\beta, [Y_\alpha, H]]\right) \\ &= \exp\left(\omega(H)z + \frac{1}{2} \sum_{\alpha+\beta=\omega} \alpha(H) c_{\alpha, \beta} y_\alpha y_\beta\right) tZ \dots \end{aligned}$$

where again the only relevant component is the linear term in  $t$  on  $Z$ . Therefore

$$p^H(n) = \omega(H)z - \frac{1}{2} \sum_{[\alpha] \in \Sigma_{1/2}/\sim} y_\alpha y_{\omega-\alpha} ((\omega - \alpha)(H) - \alpha(H)) c_{\alpha, \omega-\alpha},$$

as required.

(iii) Since  $[Z, \mathfrak{n}] = 0$ , the conclusion is obvious.  $\square$

PROPOSITION 4. *Let  $\nu$  be either in  $\Sigma_0$  or in  $-\Sigma_0$ . Let  $\mathcal{A} = \{\alpha \in \Sigma_{1/2} : \nu + \alpha \in \Sigma\}$ , and write  $\mathcal{A} = \mathcal{A}_+ \cup \mathcal{A}_\neq$ , where  $\mathcal{A}_\neq = \{\alpha \in \mathcal{A} : \alpha \neq \omega - (\nu + \alpha)\}$*

and  $\mathcal{A}_\neq = \{\alpha \in \mathcal{A} : \alpha \notin \mathcal{A}_\neq\}$ . Then

$$p^\nu(n) = \sum_{\alpha \in \mathcal{A}_\neq} \frac{c_{\alpha,\nu}}{c_{\alpha,\omega-\alpha}} p^{\nu+\alpha}(n) p^{\omega-\alpha}(n) + \frac{1}{2} \sum_{\alpha \in \mathcal{A}_=} \frac{c_{\alpha,\nu}}{c_{\alpha,\omega-\alpha}} [p^{\omega-\alpha}(n)]^2.$$

PROOF. First recall that if  $\alpha \in \mathcal{A}$ , then

$$(\nu + \alpha, \omega) = (\nu, \omega) + (\alpha, \omega) = \frac{1}{2}(\omega, \omega),$$

whence  $\nu + \alpha \in \Sigma_{1/2}$ . Moreover, by definition  $\omega + \nu$  cannot be a root even if  $\nu$  is negative. Therefore, proceeding as in the previous proposition, we obtain

$$\begin{aligned} n_{1/2}^{-1} n_1^{-1} \exp(tX_\nu) n_1 n_{1/2} &= n_{1/2}^{-1} \exp(tX_\nu) n_{1/2} \\ &= \exp(tX_\nu - t \sum_{\alpha \in \Sigma_{1/2}} y_\alpha [Y_\alpha, X_\nu] + \frac{t}{2} \sum_{\alpha_1, \alpha_2 \in \Sigma_{1/2}} y_{\alpha_1} y_{\alpha_2} [Y_{\alpha_2}, [Y_{\alpha_1}, X_\nu]]) \\ &= \exp\left(\frac{t}{2} \sum_{\nu + \alpha_1 + \alpha_2 = \omega} c_{\alpha_1, \nu} c_{\alpha_2, \nu + \alpha_1} y_{\alpha_1} y_{\alpha_2} Z\right) \exp\left(tX_\nu - t \sum_{\alpha \in A} c_{\alpha, \nu} y_\alpha Y_{\alpha + \nu}\right) \end{aligned}$$

Again by (18),

$$p^\nu(n) = \frac{1}{2} \sum_{\nu + \alpha_1 + \alpha_2 = \omega} c_{\alpha_1, \nu} c_{\alpha_2, \nu + \alpha_1} y_{\alpha_1} y_{\alpha_2},$$

where  $\alpha_1$  and  $\alpha_2$  are in  $\Sigma_{1/2}$ .

We now notice that if  $\alpha_1 \neq \alpha_2$ , then both  $\nu + \alpha_1 + \alpha_2$  and  $\nu + \alpha_2 + \alpha_1$  are  $\omega$ -chains. Thus the coefficient of the monomial  $y_{\alpha_1} y_{\alpha_2}$  is  $c_{\alpha_1, \nu} c_{\alpha_2, \nu + \alpha_1} + c_{\alpha_2, \nu} c_{\alpha_1, \nu + \alpha_2}$ . Furthermore, since  $\nu + \alpha_1 + \alpha_2 = \omega$ , the root  $\alpha_2$  must be equal to  $\omega - (\nu + \alpha_1)$ . Using the Jacobi identity:

$$[X_\alpha, [X_{\omega-(\nu+\alpha)}, X_\nu]] = [X_{\omega-(\nu+\alpha)}, [X_\alpha, X_\nu]] + 0$$

i.e.

$$c_{\omega-(\nu+\alpha), \nu} c_{\alpha, \omega-\alpha} = c_{\alpha, \nu} c_{\omega-(\nu+\alpha), \nu+\alpha}.$$

So, by (i) of Proposition 3, we can write  $p^\nu$  as follows

$$p^\nu(n) = \sum_{\alpha \in \mathcal{A}_\neq} \frac{c_{\alpha,\nu}}{c_{\alpha,\omega-\alpha}} p^{\nu+\alpha}(n) p^{\omega-\alpha}(n) + \frac{1}{2} \sum_{\alpha \in \mathcal{A}_=} \frac{c_{\alpha,\nu}}{c_{\alpha,\omega-\alpha}} [p^{\omega-\alpha}(n)]^2.$$

□

Since  $[X_\alpha, X_{-\alpha}] = B(X_\alpha, X_{-\alpha})H_\alpha$ , by suitably normalizing the basis vectors, one may assume that  $B(X_\alpha, X_{-\alpha}) = 1$ . In order to semplify the

notations, from now on we fix such a basis. Then the following relations for the structure constants hold (see [21], Sec.1, Chap.VI):

$$(19) \quad \begin{aligned} c_{\alpha,\beta} &= c_{\beta,\gamma} = c_{\gamma,\alpha}, \\ c_{-\alpha,\alpha+\beta} &= c_{\alpha,\beta}, \end{aligned}$$

for every  $\alpha, \beta, \gamma \in \Sigma$  s.t.  $\alpha + \beta + \gamma = 0$ .

**PROPOSITION 5.** *If  $\gamma \in \Sigma_{1/2}$ , then*

$$p^{-\gamma}(n) = -\frac{1}{c_{\gamma,\omega-\gamma}} p^{\omega-\gamma}(n) p^{H(\gamma)}(n) + \frac{1}{3} \sum_{\alpha \in \Sigma_{1/2}; -\gamma + \alpha \in \pm \Sigma_0} \frac{c_{\alpha,-\gamma}}{c_{\alpha,\omega-\alpha}} p^{\omega-\alpha}(n) p^{\alpha-\gamma}(n),$$

where  $H = H(\gamma) \in \mathfrak{a}$  is the solution of the linear system

$$(20) \quad \begin{cases} \omega(H) = -\omega(H_\gamma) \\ (3\alpha - \omega)(H) = -\alpha(H_\gamma) \end{cases} \quad \forall \alpha \in \tilde{\Sigma}_{1/2}.$$

Furthermore, let  $\gamma$  be a simple root and  $H(\gamma)$  the corresponding solution of (41). If  $\gamma' = \gamma + \delta_1 + \delta_2 + \dots + \delta_p$ , for some simple roots  $\delta_1, \dots, \delta_p \in \Sigma_0$ , then

$$(21) \quad H(\gamma') = H(\gamma) - \frac{1}{3}(H_{\delta_1} + \dots + H_{\delta_p}).$$

**PROOF.** As before

$$\begin{aligned} n_{1/2}^{-1} n_1^{-1} \exp(tY_{-\gamma}) n_1 n_{1/2} &= n_{1/2}^{-1} \exp(tY_{-\gamma} - tc_{\omega,-\gamma} z Y_{\omega-\gamma}) n_{1/2} \\ &= \exp(tY_{-\gamma} - tc_{\omega,-\gamma} z Y_{\omega-\gamma} - t \sum_{\alpha \in \Sigma_{1/2}} y_\alpha [Y_\alpha, Y_{-\gamma}] \\ &\quad + tz \sum_{\alpha \in \Sigma_{1/2}} c_{\omega,-\gamma} y_\alpha [Y_\alpha, Y_{\omega-\gamma}] \\ &\quad + \frac{t}{2} \sum_{\alpha_1, \alpha_2 \in \Sigma_{1/2}} y_{\alpha_1} y_{\alpha_2} [Y_{\alpha_2}, [Y_{\alpha_1}, Y_{-\gamma}]] \\ &\quad - \frac{t}{6} \sum_{\alpha_1, \alpha_2, \alpha_3 \in \Sigma_{1/2}} y_{\alpha_1} y_{\alpha_2} y_{\alpha_3} [Y_{\alpha_3}, [Y_{\alpha_2}, [Y_{\alpha_1}, Y_{-\gamma}]]]). \end{aligned}$$

Since  $(-\gamma + \alpha, \omega) = -(\gamma, \omega) + (\alpha, \omega) = -\frac{1}{2}(\omega, \omega) + \frac{1}{2}(\omega, \omega) = 0$  for every  $\alpha \in \Sigma_{1/2}$ , it follows that  $-\gamma + \alpha$  is either in  $\pm \Sigma_0$  or 0 or not a root. This implies that the bracket  $[Y_{\alpha_1}, Y_{-\gamma}]$  is respectively in  $\mathfrak{n}_{(0)}$ ,  $\mathfrak{a}$  or zero. Then,

by (18) we have

$$\begin{aligned} p^{-\gamma}(n) = & -\omega(H_\gamma)y_\gamma z + \frac{1}{6} \sum_{\alpha \in \Sigma_{1/2}} \alpha(H_\gamma)c_{\omega-\alpha, \alpha}y_\gamma y_\alpha y_{\omega-\alpha} \\ & - \frac{1}{6} \sum_{-\gamma+\alpha_1+\alpha_2+\alpha_3=\omega} c_{\alpha_1, -\gamma}c_{\alpha_2, -\gamma+\alpha_1}c_{\alpha_3, -\gamma+\alpha_1+\alpha_2}y_{\alpha_1}y_{\alpha_2}y_{\alpha_3}, \end{aligned}$$

with  $\alpha_1, \alpha_2, \alpha_3 \in \Sigma_{1/2}$ , and where we used that  $c_{\omega, -\gamma}c_{\gamma, \omega-\gamma} = -\omega(H_\gamma)$  (Jacobi identity). In particular, if  $-\gamma + \alpha_1 \in \pm\Sigma_0$ , by Proposition 4 we can write the factor  $y_{\alpha_2}y_{\alpha_3}(c_{\alpha_2, -\gamma+\alpha_1}c_{\alpha_3, -\gamma+\alpha_1+\alpha_2})$  as  $2p^{-\gamma+\alpha_1}(n)$ . Moreover, by (i) of Proposition 3,  $y_{\alpha_1} = -\frac{p^{\omega-\alpha_1}(n)}{c_{\alpha_1, \omega-\alpha_1}}$  and  $y_\gamma = -\frac{p^{\omega-\gamma}(n)}{c_{\gamma, \omega-\gamma}}$ . Therefore

$$\begin{aligned} p^{-\gamma}(n) = & -\frac{p^{\omega-\gamma}(n)}{c_{\gamma, \omega-\gamma}} \left\{ -\omega(H_\gamma)z \right. \\ & + \frac{1}{6} \sum_{\alpha \in \Sigma_{1/2}/\sim} c_{\alpha, \omega-\alpha}y_\alpha y_{\omega-\alpha}((\omega - \alpha)(H_\gamma) - \alpha(H_\gamma)) \left. \right\} \\ & + \frac{1}{3} \sum_{\alpha \in \Sigma_{1/2}/-\gamma+\alpha \in \pm\Sigma_0} \frac{c_{\alpha, -\gamma}}{c_{\alpha, \omega-\alpha}} p^{\omega-\alpha}(n)p^{-\gamma+\alpha}(n). \end{aligned}$$

Consider the polynomial in curly braces and compare it with (ii) of Proposition 3. We deduce that it has the form of a polynomial corresponding to some element  $H \in \mathfrak{a}$ , provided that  $H$  satisfies

$$\begin{cases} \omega(H) = -\omega(H_\gamma) \\ -\frac{1}{2}c_{\alpha, \omega-\alpha}((\omega - \alpha)(H) - \alpha(H)) = \frac{1}{6}c_{\alpha, \omega-\alpha}((\omega - \alpha)(H_\gamma) - \alpha(H_\gamma)), \end{cases}$$

for every  $\alpha \in \tilde{\Sigma}_{1/2}$ . This system is equivalent to

$$\begin{cases} \omega(H) = -\omega(H_\gamma) \\ (3\alpha - \omega)(H) = -\alpha(H_\gamma) \quad \forall \alpha \in \tilde{\Sigma}_{1/2}. \end{cases}$$

In order to conclude the proof of the first statement, it is enough to prove that this linear system has a solution. To this end, take a maximal set of linear independent vectors in  $\tilde{\Sigma}_{1/2}$ , and denote it by  $\mathcal{B}(\tilde{\Sigma}_{1/2})$ . Since the restricted roots generate a vector space of dimension equal to the rank of  $\mathfrak{g}$ , say  $l$ , there are at most  $l$  elements in  $\mathcal{B}(\tilde{\Sigma}_{1/2})$ . By inspection of the complete Dinkin diagrams of split semisimple Lie algebras ([5]) one sees that the cardinality of  $\mathcal{B}(\tilde{\Sigma}_{1/2})$  is at least  $l-1$ . Hence there are two possible cases.

(a)  $\#\mathcal{B}(\tilde{\Sigma}_{1/2}) = l - 1$ , say  $\mathcal{B}(\tilde{\Sigma}_{1/2}) = \{\alpha_1, \dots, \alpha_{l-1}\}$ . In this case  $\beta = \sum_{i=1}^{l-1} a_i \alpha_i$  for every  $\beta \in \tilde{\Sigma}_{1/2}$ . Since  $\beta \in \Sigma_{1/2}$ , its inner product with  $\omega$  is half of the square norm of  $\omega$ , hence  $\frac{1}{2}(\omega, \omega) = (\sum_{i=1}^{l-1} a_i \alpha_i, \omega) = \sum_{i=1}^{l-1} a_i (\alpha_i, \omega) = \sum_{i=1}^{l-1} a_i \frac{1}{2}(\omega, \omega)$ , which implies  $\sum_{i=1}^{l-1} a_i = 1$ . If  $H$  is a solution of the subsystem of (20)

$$(22) \quad \begin{cases} \omega(H) = -\omega(H_\gamma) \\ (3\alpha_i - \omega)(H) = -\alpha_i(H_\gamma) \quad \forall i = 1 \dots l-1, \end{cases}$$

then  $H$  solves also (41). Indeed

$$(3\beta - \omega)(H) = \left( 3 \sum_{i=1}^{l-1} a_i \alpha_i - \sum_{i=1}^{l-1} a_i \omega \right) (H) = - \sum_{i=1}^{l-1} a_i \alpha_i (H_\gamma) = -\beta(H_\gamma).$$

The linear system (22) has  $l$  equations and  $l$  variables, and the associated matrix is diagonal. Therefore it has exactly one solution.

(b)  $\#\mathcal{B}(\tilde{\Sigma}_{1/2}) = l$ , so that  $\omega = \sum_{i=1}^l b_i \alpha_i$ , that is  $(\omega, \omega) = \sum_{i=1}^l b_i (\alpha_i, \omega) = \sum_{i=1}^l b_i \frac{1}{2}(\omega, \omega)$ . This implies that  $\sum_{i=1}^l b_i = 2$ . It turns out that any equation in (20) depends linearly on the set of equations

$$(3\alpha_i - \omega)(H) = -\alpha_i(H_\gamma) \quad \forall i = 1 \dots l.$$

Indeed

$$-\omega(H_\gamma) = - \sum_{i=1}^l b_i \alpha_i (H_\gamma) = \left( 3 \sum_{i=1}^l b_i \alpha_i (H) - \sum_{i=1}^l b_i \omega \right) (H) = \omega(H),$$

and

$$(3\beta - \omega)(H) = \left( 3 \sum_{i=1}^l a_i \alpha_i - \sum_{i=1}^l a_i \omega \right) (H) = - \sum_{i=1}^l a_i \alpha_i (H_\gamma) = -\beta(H_\gamma),$$

where  $\beta = \sum_{i=1}^l a_i \alpha_i$  is any root in  $\tilde{\Sigma}_{1/2}$ .

Let now  $\gamma \in \Delta_{1/2}$ , and let  $H(\gamma)$  be the corresponding solution of (20). Let  $\gamma'$  be another root in  $\Sigma_{1/2}$  and  $\delta_1, \dots, \delta_p$  simple roots such that  $\gamma' = \gamma + \delta_1 + \dots + \delta_p$ . It is an easy calculation to check that  $H(\gamma') = H(\gamma) - \frac{1}{3}(H_{\delta_1} + \dots + H_{\delta_p})$ .  $\square$

**Remarks.** From the theory of root systems it follows that every root  $\gamma'$  in  $\Sigma_{1/2}$  can be written as  $\gamma + \delta_1 + \dots + \delta_p$ , with  $\gamma \in \Delta_{1/2}$  and for some  $\delta_1, \dots, \delta_p \in \Delta_0$  (see, e.g., Lemma 3.5 in [12]). This fact, together with (21), tells us that we must solve (20) only for  $\gamma \in \Delta_{1/2}$ . Furthermore, by the classification of root systems, we know that there exists exactly one

simple root belonging to  $\Sigma_{1/2}$ , except the case of the root system  $A_n$ , for which  $\Sigma_{1/2}$  consists of two roots [5].

**PROPOSITION 6.** *Fix  $H = \frac{1}{\sqrt{2\omega(H_\omega)}}H_\omega$ . Then*

$$p^{-\omega}(n) = -(p^H(n))^2 - \frac{1}{4} \sum_{\alpha \in \Sigma_{1/2}} p^{\omega-\alpha}(n)p^{\alpha-\omega}(n).$$

**PROOF.** Notice that in order to complete  $-\omega$  to an  $\omega$ -chain we need exactly four roots in  $\Sigma_{1/2}$ . We obtain

$$\begin{aligned} & n_{1/2}^{-1} n_1^{-1} \exp tX_{-\omega} n_1 n_{1/2} \\ &= n_{1/2}^{-1} \exp \left( tX_{-\omega} - tzH_\omega - \frac{t}{2}z^2\omega(H_\omega)Z \right) n_{1/2} \\ &= \exp \left( -\frac{t}{2}z^2\omega(H_\omega)Z \right) \times \\ & \quad \times \exp \left( tX_{-\omega} - tzH_\omega - t \sum_{\alpha \in \Sigma_{1/2}} c_{\alpha, -\omega} y_\alpha Y_{-\omega+\alpha} \right. \\ & \quad \left. - tz \sum_{\alpha \in \Sigma_{1/2}} \alpha(H_\omega) y_\alpha Y_\alpha + \frac{t}{2} \sum_{\alpha_1, \alpha_2 \in \Sigma_{1/2}} y_{\alpha_1} y_{\alpha_2} [Y_{\alpha_2}, [Y_{\alpha_1}, X_{-\omega}]] \right. \\ & \quad \left. + \frac{t}{2}z \sum_{\alpha \in \Sigma_{1/2}} c_{\omega-\alpha, \alpha} \alpha(H_\omega) y_\alpha y_{\omega-\alpha} Z \right. \\ & \quad \left. - \frac{t}{6} \sum_{\alpha_1, \alpha_2, \alpha_3 \in \Sigma_{1/2}} y_{\alpha_1} y_{\alpha_2} y_{\alpha_3} [Y_{\alpha_3}, [Y_{\alpha_2}, [Y_{\alpha_1}, X_{-\omega}]]] \right. \\ & \quad \left. + \frac{t}{24} \sum_{\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \Sigma_{1/2}} y_{\alpha_1} y_{\alpha_2} y_{\alpha_3} y_{\alpha_4} [Y_{\alpha_4}, [Y_{\alpha_3}, [Y_{\alpha_2}, [Y_{\alpha_1}, X_{-\omega}]]]] \right) \\ &= \exp \left( \left( -\frac{t}{2}z^2\omega(H_\omega) + \frac{t}{2}z \sum_{\alpha \in \Sigma_{1/2}} c_{\omega-\alpha, \alpha} \alpha(H_\omega) y_\alpha y_{\omega-\alpha} \right. \right. \\ & \quad \left. \left. + \frac{t}{24} \sum_{\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \Sigma_{1/2}} c_{\alpha_1, -\omega} c_{\alpha_2, -\omega+\alpha_1} \times \right. \right. \\ & \quad \left. \left. \times c_{\alpha_3, -\omega+\alpha_1+\alpha_2} c_{\alpha_4, -\omega+\alpha_1+\alpha_2+\alpha_3} y_{\alpha_1} y_{\alpha_2} y_{\alpha_3} y_{\alpha_4} \right) Z \right) \dots \end{aligned}$$

Writing the corresponding polynomial, we can split the last summand of the above formula according to the fact that the chains  $-\omega+\alpha_1+\alpha_2+\alpha_3+\alpha_4$  are of two kinds. In fact we have either  $\alpha_2 = \omega - \alpha_1$ , so that  $-\omega + \alpha_1 + \alpha_2 = 0$ ,

or  $-\omega + \alpha_1 + \alpha_2 \in \pm\Sigma_0$ . Therefore

$$\begin{aligned}
(23) \quad p^{-\omega}(n) = & -\frac{1}{2}z^2\omega(H_\omega) \\
& + \frac{1}{2}z \sum_{\alpha \in \Sigma_{1/2}} c_{\omega-\alpha, \alpha} \alpha(H_\omega) y_\alpha y_{\omega-\alpha} \\
& - \frac{1}{24} \sum_{\alpha, \beta \in \Sigma_{1/2}} c_{\alpha, -\omega} c_{\omega-\beta, \beta} \beta(H_{\omega-\alpha}) y_\alpha y_{\omega-\alpha} y_\beta y_{\omega-\beta} \\
& + \frac{1}{24} \sum_{\alpha_i \in \Sigma_{1/2}: -\omega + \alpha_1 + \alpha_2 \in \pm\Sigma_0} c_{\alpha_1, -\omega} c_{\alpha_2, -\omega + \alpha_1} \times \\
& \quad \times c_{\alpha_3, -\omega + \alpha_1 + \alpha_2} c_{\alpha_4, -\omega + \alpha_1 + \alpha_2 + \alpha_3} y_{\alpha_1} y_{\alpha_2} y_{\alpha_3} y_{\alpha_4}.
\end{aligned}$$

Since for every  $\alpha \in \Sigma_{1/2}$

$$B(H_\omega, H) = \omega(H) = (\omega - \alpha)(H) + \alpha(H) = B(H_{\omega-\alpha}, H) + B(H_\alpha, H),$$

it follows that  $H_\omega = H_{\omega-\alpha} + H_\alpha$ . Moreover,

$$\begin{aligned}
\sum_{\alpha \in \Sigma_{1/2}} c_{\omega-\alpha, \alpha} \alpha(H_\omega) y_\alpha y_{\omega-\alpha} &= \sum_{\alpha \in \Sigma_{1/2}} c_{\omega-\alpha, \alpha} \frac{\omega(H_\omega)}{2} y_\alpha y_{\omega-\alpha} \\
&= \frac{\omega(H_\omega)}{2} \sum_{\alpha \in \Sigma_{1/2}} c_{\omega-\alpha, \alpha} y_\alpha y_{\omega-\alpha} \\
&= 0,
\end{aligned}$$

because

$$\sum_{\alpha \in \Sigma_{1/2}} c_{\omega-\alpha, \alpha} y_\alpha y_{\omega-\alpha} = \sum_{\alpha \in \tilde{\Sigma}_{1/2}} (c_{\omega-\alpha, \alpha} - c_{\omega-\alpha, \alpha}) y_\alpha y_{\omega-\alpha}.$$

Comparing with the polynomial formula of Proposition 5, and observing that (19) implies

$$c_{\omega-\alpha, \alpha} = c_{\alpha, -\omega},$$

the sum in (23) becomes

$$\begin{aligned}
& \frac{1}{2}z \sum_{\alpha \in \Sigma_{1/2}} c_{\omega-\alpha, \alpha} \alpha(H_\omega) y_\alpha y_{\omega-\alpha} \\
& - \frac{1}{24} \sum_{\alpha, \beta \in \Sigma_{1/2}} c_{\alpha, -\omega} c_{\omega-\beta, \beta} \beta(H_{\omega-\alpha}) y_\alpha y_{\omega-\alpha} y_\beta y_{\omega-\beta} \\
& + \frac{1}{24} \sum_{\alpha_i \in \Sigma_{1/2}: -\omega+\alpha_1+\alpha_2 \in \pm \Sigma_0} c_{\alpha_1, -\omega} c_{\alpha_2, -\omega+\alpha_1} \times \\
& \quad \times c_{\alpha_3, -\omega+\alpha_1+\alpha_2} c_{\alpha_4, -\omega+\alpha_1+\alpha_2+\alpha_3} y_{\alpha_1} y_{\alpha_2} y_{\alpha_3} y_{\alpha_4} \\
& = -\frac{1}{4} \left\{ \sum_{\alpha \in \Sigma_{1/2}} c_{\alpha, -\omega} y_\alpha (-\omega(H_{\omega-\alpha}) z y_{\omega-\alpha} \right. \\
& \quad \left. + \frac{1}{6} y_{\omega-\alpha} \sum_{\beta \in \Sigma_{1/2}} c_{\omega-\beta, \beta} \beta(H_{\omega-\alpha}) y_\beta y_{\omega-\beta} \right. \\
& \quad \left. - \frac{1}{6} \sum_{\alpha_i \in \Sigma_{1/2}: -\omega+\alpha+\alpha_1 \in \pm \Sigma_0} c_{\alpha_1, -\omega+\alpha} c_{\alpha_2, -\omega+\alpha+\alpha_1} c_{\alpha_3, -\omega+\alpha+\alpha_1+\alpha_2} y_{\alpha_1} y_{\alpha_2} y_{\alpha_3} \right\} \\
& = -\frac{1}{4} \sum_{\alpha \in \Sigma_{1/2}} p^{\omega-\alpha}(n) p^{\alpha-\omega}(n),
\end{aligned}$$

where the last equality follows by (i) of Proposition 3 and Proposition 5. Finally, by Proposition 4,  $\frac{z^2}{2}\omega(H_\omega) = p^H(n)p^H(n)$  if  $H$  satisfies

$$\begin{cases} \omega(H) - 2\alpha(H) = 0 \\ \omega(H) = \sqrt{\frac{\omega(H_\omega)}{2}}, \end{cases}$$

i.e.

$$\begin{cases} \alpha(H) = \frac{1}{2}\omega(H) \\ \omega(H) = \sqrt{\frac{\omega(H_\omega)}{2}}. \end{cases}$$

It is a simple calculation to verify that  $H = \frac{1}{\sqrt{2\omega(H_\omega)}} H_\omega$  satisfies the equations above. We then conclude that

$$p^{-\omega}(n) = -(p^H(n))^2 - \frac{1}{4} \sum_{\alpha \in \Sigma_{1/2}} p^{\omega-\alpha}(n) p^{\alpha-\omega}(n),$$

as required.  $\square$



## CHAPTER 3

### **Hessenberg manifolds**

The Hessenberg manifolds arise as a natural class of submanifolds of the spaces  $G/P$ . A crucial point of the present work is to investigate in detail the stratification of the tangent bundle of these manifolds. We shall see that they inherit from  $G/P$  a structure that allows us to define the appropriate version of multicontact mapping. In the first section we define the classical Hessenberg manifolds, viewed as submanifolds of the complete flag manifolds. In the second section we define them in the more general context of  $G/P$ , where  $G$  is a real semisimple Lie group and  $P$  is a minimal parabolic subgroup of  $G$ . The results and definitions that are considered in the second section are taken from [14].

#### **1. The basic context: the Hessenberg flags**

We collect some notions about the Hessenberg flag manifolds. For further details, see [13]. The presentation that follows differs somewhat from the standard version outlined in the introduction. The main reason for doing so is that our natural (local) environment is the Iwasawa nilpotent group that in the standard setting would be  $\bar{N}$ , a lower triangular group. We find it more natural to be working on  $N$ , the unipotent upper triangular group.

Let  $G = \mathrm{SL}(n, \mathbb{R})$ , and let  $P$  be its minimal parabolic subgroup given by the lower triangular matrices. The homogeneous space obtained by the quotient  $G/P$  realizes the complete *flag manifold*, in the following sense. Define

$$\mathrm{Flag}(n) = \{(S_1, \dots, S_{n-1}) : S_1 \subset \dots \subset S_{n-1}\},$$

where each  $S_k$  is a subspace of  $\mathbb{R}^n$  such that  $\dim S_k = k$ . The set  $\mathrm{Flag}(n)$  is a smooth and compact manifold, called complete flag manifold. It is easy to check that  $\mathrm{SL}(n, \mathbb{R})$  acts in a transitive way on  $\mathrm{Flag}(n)$  by the natural action

$$g : (S_1, \dots, S_{n-1}) \mapsto (gS_1, \dots, gS_{n-1}),$$

for every  $g \in \mathrm{SL}(n, \mathbb{R})$ . We can view any flag as a matrix of column vectors

$$(24) \quad \begin{bmatrix} v_n & \cdots & v_1 \end{bmatrix}$$

where  $S_1 = \mathrm{span}(v_1)$ ,  $S_2 = \mathrm{span}(v_1, v_2), \dots, \mathbb{R}^n = \mathrm{span}(v_1, v_2, \dots, v_n)$ . Consider the flag corresponding to the identity matrix. The isotropy group at this flag is the group  $P$  of lower triangular matrices, so that

$$\mathrm{Flag}(n) = G/P.$$

Each matrix  $A \in \mathrm{SL}(n, \mathbb{R})$  representing a flag can be reduced to a unipotent upper triangular matrix by changing the representative in  $G/P$ . This is done by the Gauss algorithm, provided we restrict ourselves to an open and dense subset of  $G/P$  where we assume that some entries are not zero in order to make the Gauss algorithm work. Summarizing, the nilpotent subgroup of  $\mathrm{SL}(n, \mathbb{R})$  defined by the unipotent upper triangular matrices and denoted by  $N$  is identified with an open and dense subset of  $\mathrm{Flag}(n)$ .

Consider now the Lie algebra  $\mathfrak{sl}(n, \mathbb{R})$  given by the matrices whose trace is zero, and take a diagonal matrix  $H$  in  $\mathfrak{sl}(n, \mathbb{R})$  with distinct entries and non-zero determinant. Furthermore, fix an integer  $p$  such that  $1 \leq p < n - 1$ . We say that a flag  $(S_1, \dots, S_{n-1})$  is a *type p Hessenberg flag for  $H$*  if the matrix  $H$  shifts the linear space  $S_i$  within  $S_{i+p}$ , that is

$$HS_i \subset S_{i+p}.$$

The set of all such flags is a smooth manifold that we denote by  $\mathrm{Hess}_p(H)$  and call *p-th Hessenberg manifold*. By considering again the flag manifold as the homogeneous space  $G/P$ , a flag  $X \in G/P$  is a type  $p$  Hessenberg flag for  $H$  if and only if it satisfies

$$(25) \quad HX = XR,$$

where  $R = (r_{i,j})$  is a matrix in  $\mathfrak{sl}(n, \mathbb{R})$  such that

$$r_{i,j} = \begin{cases} 0 & \text{if } (i, j) = (i, p+i+1) \text{ or } (i, j) = (p+j+1, j) \\ * & \text{otherwise,} \end{cases}$$

i.e.  $R$  is a matrix where all entries above the  $p+1$ -th diagonal are zero. In order to see that (25) is equivalent to the definition of a Hessenberg flag it is enough to visualize  $X$  as a matrix of column vectors as in (24), and to notice that the product  $XR$  maps the last column, that corresponds to  $S_1$ , in a linear combination of the last  $p+1$ -th, and so on. Since  $X$  is invertible, we can rewrite (25) as

$$(26) \quad X^{-1}HX = R.$$

The formula above can be used for computing local algebraic equations for  $\text{Hess}_p(H)$ . We restrict ourselves to the dense subset  $N$  of upper unipotent triangular matrices, and we write  $X \in N$  as  $(x_{i,j})_{i,j}$ , with  $i < j$ . Moreover, write  $H = (\lambda_i)_i$ . Then we compute the product  $X^{-1}HX$  and we put equal to zero the entries above the  $p+1$ -th diagonal, obtaining

$$(27) \quad f_{i,j}(X) = (\lambda_j - \lambda_i)x_{i,j} + \sum_{t=1}^{j-i-1} \sum_{i < k_1 < \dots < k_t < j} (-1)^t (\lambda_{k_t} - \lambda_j)x_{i,k_1}x_{k_1,k_2} \cdots x_{k_t,j} = 0,$$

for every pair  $(i, j)$ ,  $i < j$ . Thus we have a set of equations defining the Hessenberg manifold in a dense subset.

Notice that in the formula above, the coefficients  $\lambda_j - \lambda_i$  and  $\lambda_{k_t} - \lambda_j$  are always non-zero, because the entries of  $H$  are all distinct. This implies that the Jacobian matrix associated with the set of equations (27) has maximal rank, so that  $\text{Hess}_p(\mathbb{R})$  is smooth.

We conclude observing that the formula (26) continues to make sense if the matrix  $R$  is taken in a set which is closed under conjugation by elements in  $P$ . Hence we can abstractly give a more general definition of Hessenberg manifold by considering matrices of the form

$$R = \begin{array}{c} \boxed{\begin{matrix} * & & & & \\ * & * & * & & \\ * & * & * & & \\ * & * & & & \\ * & & & & \\ & & & & 0 \end{matrix}} \\ . \end{array}$$

By this remark one generalizes the definition of Hessenberg manifolds to the context of semisimple Lie group, as we show in the next section.

## 2. Real Hessenberg manifolds

We report the definition and some properties of real Hessenberg manifolds, that can be found in [14]. Let  $G$  be a connected (real) semisimple noncompact Lie group with finite center. Let  $\mathfrak{g}$  be its Lie algebra and  $\Sigma$  the corresponding restricted root system. Choose an ordering in  $\Sigma$ , fix the set

of positive roots  $\Sigma_+$  and the set of positive simple roots  $\Delta = \{\delta_1, \dots, \delta_l\}$ . Let  $\mathcal{R}$  be some proper subset of the set of the positive roots  $\Sigma_+$ . We call it of Hessenberg type if it satisfies the following property:

if  $\alpha \in \mathcal{R}$  and  $\beta$  is any negative root such that  $\alpha + \beta \in \Sigma_+$ , then  $\alpha + \beta \in \mathcal{R}$ .

Denote by  $\mathcal{C}$  the complement in  $\Sigma_+$  of  $\mathcal{R}$ . Let  $P$  be the minimal parabolic subgroup of  $G$ . We define the Hessenberg manifold corresponding to  $\mathcal{R}$  and to some regular element  $H$  in the Cartan subspace  $\mathfrak{a}$  as the following submanifold of  $G/P$ :

$$\text{Hess}_{\mathcal{R}}(H) = \{\langle g \rangle_P \in G/P : \text{Ad } g^{-1}H \in \mathfrak{b}_{\mathcal{R}}\},$$

where  $\mathfrak{b}_{\mathcal{R}} = \mathfrak{a} \oplus \mathfrak{n} \oplus \bigoplus_{\gamma \in \mathcal{R}} \mathfrak{g}_{\gamma}$ . The Hessenberg manifolds are smooth submanifolds of  $G/P$ , and they are algebraic varieties. The next proposition can be found in [DP], but we recall the proof because it shows the explicit algebraic equations locally defining  $\text{Hess}_{\mathcal{R}}(H)$ . The notations are those that we introduced in Ch. 1. We just remind that  $W$  denotes the Weyl group and  $\text{MAN}$  the Langlands decomposition of  $P$ .

**PROPOSITION 7.** ([14])  $\text{Hess}_{\mathcal{R}}(H)$  is a smooth submanifold of  $G/P$  of dimension  $\sum_{\alpha \in \mathcal{R}} m_{\alpha}$ .

We prove the smoothness of  $\text{Hess}_{\mathcal{R}}(H)$  by writing local defining equations. The local coordinates are given by the “Bruhat charts” in  $G/P$ . It is useful in this context to recall in some details the Bruhat decomposition of  $G$  and  $G/P$ , in order to construct a somewhat canonical open covering of  $G/P$ . The Bruhat decomposition of  $G$  is the disjoint union

$$G = \coprod_{w \in W} PwP.$$

Observe that

$$PwP = \text{MAN}(wN) = \text{MA}\bar{N}(w\bar{N}w^{-1})w = \text{MA}(\bar{N}\bar{N}^w)w,$$

where  $\bar{N}^w := w\bar{N}w^{-1}$ ; therefore  $\mathcal{P} := PwP = \text{MA}(\bar{N}\bar{N}^w)w$ . In particular, if  $w_0$  is the element of the Weyl group which exchanges the negative and the positive roots, then  $\bar{N}^{w_0} = N := \exp \mathfrak{n}$ . Therefore the “big cell” is

$$\mathcal{P}_{w_0} = \text{MA}\bar{N}Nw_0 = \bar{N}\text{MA}w_0\bar{N} = w_0(\text{NMA}\bar{N}).$$

The Bruhat decomposition of  $G/P$  is trivially induced by that of  $G$ :

$$G/P = \coprod_{w \in W} \langle \mathcal{P}_w \rangle_P = \coprod_{w \in W} \overline{\mathcal{P}}_w.$$

Notice that  $\overline{\mathcal{P}}_w = \langle w_0 N \rangle_{\mathcal{P}}$ . Now we set, for  $w \in W$ ,

$$\begin{aligned}\overline{N}^{(w)} &= \overline{N} \cap w N w^{-1} = \exp(\overline{\mathfrak{n}} \cap \text{Ad } w \mathfrak{n}) \\ \overline{N}_{(w)} &= \overline{N} \cap w \overline{N} w^{-1} = \exp(\overline{\mathfrak{n}} \cap \text{Ad } w \overline{\mathfrak{n}}).\end{aligned}$$

Then  $\overline{N} = \overline{N}^{(w)} \overline{N}_{(w)} = \overline{N}_{(w)} \overline{N}^{(w)}$ .

**PROPOSITION 8.** ([14])  $\overline{NN}^w \subset w N \overline{N} w^{-1}$ .

**PROOF.** Since  $\overline{N}_{(w)} \overline{N}^w = (\overline{N} \cap w \overline{N} w^{-1}) \overline{N}^w = (\overline{N} \cap \overline{N}^w) N^w = \overline{NN}^w \cap \overline{N}^w$ , we have that

$$\begin{aligned}\overline{N}^{(w)} \overline{N}_{(w)} \overline{N}^w &= \overline{N}^{(w)} (\overline{NN}^w \cap \overline{N}^w) \\ &= (\overline{N} \cap w N w^{-1}) (\overline{NN}^w \cap \overline{N}^w) \\ &= [\overline{N} (\overline{NN}^w \cap \overline{N}^w)] \cap [w N w^{-1} (\overline{NN}^w \cap \overline{N}^w)] \\ &\subset w N w^{-1} (\overline{NN}^w \cap \overline{N}^w) \\ &\subset w N w^{-1} \overline{N}^w \\ &= w N w^{-1} w \overline{N} w^{-1} \\ &= w N \overline{N} w^{-1}.\end{aligned}$$

□

**COROLLARY 9.** ([14])  $\mathcal{P}_w \subset w N M A \overline{N}$ .

**PROOF.** Using Proposition 8, we get

$$\begin{aligned}\mathcal{P}_w &= M A (\overline{NN}^w) w \\ &\subset M A (w N \overline{N} w^{-1}) w \\ &= M A w (w_0 \overline{N} w_0) \overline{N} \\ &= w M A w_0 \overline{N} w_0 \overline{N} \\ &= w w_0 M A \overline{N} w_0 \overline{N} \\ &= w w_0 \overline{N} M A w_0 \overline{N} \\ &= w w_0 \overline{N} w_0 M A \overline{N} \\ &= w N M A \overline{N}.\end{aligned}$$

□

From this corollary it follows that the open sets

$$ch(w) := \langle w N \rangle_{\mathcal{P}},$$

give a covering of  $G/P$ , since  $ch(w) \supset \overline{\mathcal{P}}_w$ . Hence every point in  $G/P$  can be written as  $\langle wn \rangle_P$ , for some  $w \in W$ , where  $n \in N$  is unique (once  $w$  is fixed). Let  $\langle wn \rangle_P \in ch(w)$ ; then  $\langle wn \rangle_P \in \text{Hess}_{\mathcal{R}}(H)$  if and only if  $\text{Ad}(wn)^{-1}H \in \mathfrak{b}_{\mathcal{R}}$ . But  $\text{Ad}(wn)^{-1}H = \text{Ad}n^{-1}(w^{-1}H)$ , and therefore  $\langle wn \rangle_P \in \text{Hess}_{\mathcal{R}}(H) \Leftrightarrow \langle n \rangle_P \in \text{Hess}_{\mathcal{R}}(w^{-1}H)$ . Fix  $w = 1$ . Hence we must impose that  $\text{Ad}n^{-1}H \in \mathfrak{b}_{\mathcal{R}}$ , that is,

$$(\text{Ad}n^{-1}H)_{\alpha} = 0 \quad \forall \alpha \in \mathcal{C},$$

where  $(X)_{\alpha}$  denotes the component in  $\mathfrak{g}_{\alpha}$  of  $X$ . We write  $n = \exp \nu$ , with  $\nu = \sum_{\alpha \in \Sigma_+} \nu_{\alpha} \in \mathfrak{n}$ . Therefore, because of nilpotency,

$$\begin{aligned} \text{Ad}n^{-1}H &= \text{Ad} \exp(-\nu)H \\ &= e^{-\text{ad} \nu}H \\ &= H - [\nu, H] + \frac{1}{2}[\nu, [\nu, H]] + \dots \quad (\text{finite sum}) \end{aligned}$$

and, for  $\alpha \in \Sigma_+$ ,

$$(\text{Ad}n^{-1}H)_{\alpha} = \alpha(H)\nu_{\alpha} + (\text{terms containing } x_{\beta,i}, \text{ with } \text{ht}(\beta) < \text{ht}(\alpha)).$$

If the root  $\alpha$  has multiplicity  $m_{\alpha}$ , we write  $\nu_{\alpha} = \sum_{j=1}^{m_{\alpha}} x_{\alpha,j}E_{\alpha,j}$ , where  $\{E_{\alpha,j}\}$  is a basis of  $\mathfrak{g}_{\alpha}$ . This means that we are using coordinates  $\{x_{\alpha,j}\}$ , with  $\alpha \in \Sigma_+$ ,  $j = 1, \dots, m_{\alpha}$ , in the chart  $ch(1)$ . Consequently, the equations (locally) defining  $\text{Hess}_{\mathcal{R}}(H)$  are

$$(28) \quad p_{\alpha,j}(x) = 0, \quad \alpha \in \mathcal{C}, j = 1, \dots, m_{\alpha},$$

where

$$p_{\alpha,j} = \alpha(H)x_{\alpha,j} + (\text{terms containing } x_{\beta,i}, \text{ with } \text{ht}(\beta) < \text{ht}(\alpha)).$$

the components  $x_{\gamma,k}$  of vector  $x$ , with  $\gamma \in \Sigma_+$ ,  $k = 1, \dots, m_{\gamma}$ , are ordered in such a way that  $x_{\gamma_1,k_1}$  precedes  $x_{\gamma_2,k_2}$ , if  $\text{ht}(\gamma_1) < \text{ht}(\gamma_2)$ . Obviously,  $\text{Hess}_{\mathcal{R}}(H)$  is smooth if and only if the matrix

$$J = \left[ \frac{\partial p_{\alpha,j}}{\partial x_{\gamma,k}} \right]_{\alpha \in \mathcal{C}, \gamma \in \Sigma_+}$$

has maximal rank. On the other hand

$$\frac{\partial p_{\alpha,j}}{\partial x_{\gamma,k}} = \begin{cases} 0 & \text{if } \text{ht}(\gamma) \geq \text{ht}(\alpha) \text{ and } \gamma \neq \alpha \\ 0 & \text{if } \gamma = \alpha \text{ and } k \neq j \\ \alpha(H) & \text{if } \gamma = \alpha \text{ and } k = j \\ * & \text{otherwise.} \end{cases}$$

The square submatrix

$$\tilde{J} = \left[ \frac{\partial p_{\alpha,j}}{\partial x_{\gamma,k}} \right]_{\alpha,\gamma \in \mathcal{C}}$$

is lower triangular, and the blocks on the diagonal are  $\alpha(H)I_{m_\alpha}$ . Therefore

$$\det \tilde{J} = \prod_{\alpha \in \mathcal{C}} \alpha(H)^{m_\alpha} \neq 0,$$

since  $H$  is regular, thereby proving that  $\text{Hess}_{\mathcal{R}}(H)$  is smooth. Finally, we compute the dimension of  $\text{Hess}_{\mathcal{R}}(H)$ :

$$\dim \text{Hess}_{\mathcal{R}}(H) = \dim(G/P) - \text{rank } J = \sum_{\alpha \in \Sigma_+} m_\alpha - \sum_{\alpha \in \mathcal{C}} m_\alpha = \sum_{\alpha \in \mathcal{R}} m_\alpha.$$

Finally, if  $H$  is a regular element, then also  $w^{-1}H$  is so. Hence the conclusions above are true in any chart  $ch(w)$ .

**PROPOSITION 10.** *The vector space  $\mathfrak{n}_{\mathcal{C}} = \bigoplus_{\alpha \in \mathcal{C}} \mathfrak{g}_\alpha$  is an ideal in  $\mathfrak{n}$ .*

**PROOF.** Let  $\alpha \in \mathcal{C}$  and  $\beta \in \Sigma_+$ . If  $\alpha + \beta$  is a root in  $\mathcal{R}$ , then  $\alpha + \beta - \alpha \in \mathcal{R}$ , that is false. Therefore, if  $X \in \mathfrak{g}_\alpha$  and  $Y \in \mathfrak{g}_\beta$ , then  $[X, Y] \in \mathfrak{g}_{\alpha+\beta}$ , with  $\alpha + \beta \in \mathcal{C}$ .  $\square$



## CHAPTER 4

### Multicontact vector fields on Hessenberg manifolds

This is the main chapter of the thesis. First, we transfer the multicontact structure from  $G/P$  to the Hessenberg manifolds  $\text{Hess}_{\mathcal{R}}(H)$ . Then, in the second section, we lift the problem to the Lie algebra level, that is, we define the notion of multicontact vector field on  $\text{Hess}_{\mathcal{R}}(H)$ , and we consider the Lie algebra  $MC(S)$  of all multicontact vector fields on a local model  $S$  of  $\text{Hess}_{\mathcal{R}}(H)$ , the central object of our investigation. We prove that  $MC(S)$  contains canonically a quotient algebra  $\mathfrak{q}/\mathfrak{n}_{\mathcal{C}}$ . In the third section, we show that this quotient exhausts all multicontact vector fields, if some natural additional hypotheses on the Hessenberg manifolds are assumed, synthetized in the notion of Iwasawa sub-models. In the last section we consider an example that shows that  $MC(S)$  can be larger than  $\mathfrak{q}/\mathfrak{n}_{\mathcal{C}}$ , if the additional assumptions do not hold.

#### 1. A multicontact structure on Hessenberg manifolds

The Iwasawa Lie algebra  $\mathfrak{n}$  has a multistratification  $\mathfrak{n} = \sum_{\gamma \in \Sigma_+} \mathfrak{g}_\gamma$  and a stratification by height  $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \cdots \oplus \mathfrak{n}_h$ , where  $\mathfrak{n}_i$  is the direct sum of all root spaces  $\mathfrak{g}_\gamma$  that are sum of  $i$  positive simple roots, that is  $\text{ht}(\gamma) = i$ . In particular  $[\mathfrak{n}_i, \mathfrak{n}_j] \subset \mathfrak{n}_{i+j}$ . The structure of  $\mathfrak{n}$ , viewed as the tangent space to  $N$  at the identity, allows us to give generalized versions of contact mappings (see Chap.1).

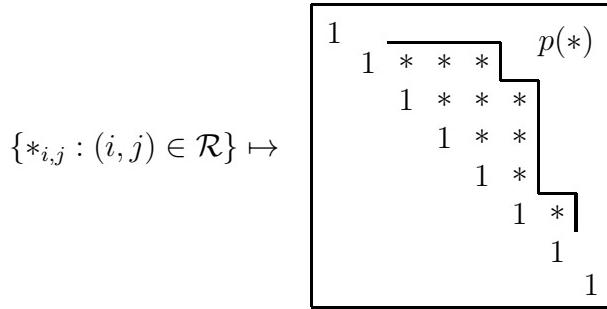
We ask ourselves how to relate with  $\mathfrak{n}$  the tangent space to some point of  $\text{Hess}_{\mathcal{R}}(H)$  because we want to transfer the stratification to the tangent bundle on some open set of  $\text{Hess}_{\mathcal{R}}(H)$ . We have seen that a choice of a Hessenberg structure  $\mathcal{R}$  determines the set of independent local coordinates  $\{x_{\alpha,j} : \alpha \in \mathcal{R}, 1 \leq j \leq m_\alpha\}$  on the Hessenberg manifold, whereas the remaining entries  $\{x_{\alpha,j} : \alpha \in \mathcal{C}, 1 \leq j \leq m_\alpha\}$  are polynomial functions of the previous ones (see (28)). The coefficients of the polynomials depend on  $H$  and more is true: those that are not zero are in fact given by functions that never vanish on the set of regular elements in  $\mathfrak{a}$ . Thus, the slice  $S$  of

$N$  obtained by setting  $x_{\alpha,j} = 0$  if  $\alpha \in \mathcal{C}$  is diffeomorphic to  $\text{Hess}_{\mathcal{R}}(H)$  for every regular element  $H$ . The graph mapping

$$(29) \quad \phi : (\{x_{\beta,k}\}_{\beta \in \mathcal{R}}, 0) \longmapsto (\{x_{\beta,k}\}_{\beta \in \mathcal{R}}, \{p_{\alpha,j}(x_{\beta,k})\}_{\alpha \in \mathcal{C}})$$

gives the diffeomorphism. For this reason we shall use  $S$  as a simplified model for  $\text{Hess}_{\mathcal{R}}(H)$ .

In  $\text{SL}(n, \mathbb{R})$ , where the nilpotent subgroup  $N$  is given by the unipotent upper triangular matrices, the diffeomorphism can be visualized as follows:



Next we define the contact structure on  $S$  and we show how it can be transferred to  $\text{Hess}_{\mathcal{R}}(H)$ . First we define the left-invariant vector fields  $X_{\alpha,j}$  on  $N$  that coincide with the partial derivative operators at the origin. Next we obtain the differentiable structure on  $S$  by considering the projections  $\bar{X}_{\alpha,j}$  on  $S$  of those vector fields that correspond to  $\alpha \in \mathcal{R}$ . Finally, the push-forwards  $\phi_*(X_{\alpha,j})$  will define the differentiable structure on the Hessenberg manifold. This structure will allow us to give a generalized version of contact mapping.

**1.1. Multicontact mappings.** Consider the basis  $\{X_{\alpha,j} : \alpha \in \Sigma_+, 1 \leq j \leq m_\alpha\}$  of left-invariant vector fields on  $N$ , where

$$X_{\alpha,j}(n) = (l_n)_{*e} \frac{\partial}{\partial x_{\alpha,j}} \Big|_e,$$

and write

$$X_{\alpha,j} = \sum_{\gamma \in \Sigma_+} \sum_{k=1}^{m_\gamma} a_{\gamma,k}^{\alpha,j} \frac{\partial}{\partial x_{\gamma,k}},$$

where  $a_{\gamma,k}^{\alpha,j}$  are some smooth functions on  $N$ . We want to compute their explicit expressions in exponential coordinates. In order to do this, we recall a result that can be found in [12]. If  $\alpha = \sum_{\delta \in \Delta} a_\delta \delta$  and  $\beta = \sum_{\delta \in \Delta} b_\delta \delta$

are two positive roots, we write  $\alpha \preceq \beta$  if  $a_\delta \leq b_\delta$  for all  $\delta \in \Delta$ . We say that  $\alpha_1 + \cdots + \alpha_n$  is a *chain* if each  $\alpha_j$  and each partial sum  $\alpha_1 + \cdots + \alpha_j$  is a root for all  $j = 1, \dots, n$ . Ordered pairs of roots can be joined by chains:

LEMMA 11 ([12]). *Let  $\alpha$  and  $\beta$  be distinct positive roots and suppose that  $\alpha \succeq \beta$ . Then there exist simple roots  $\delta_1, \dots, \delta_p$  such that  $\alpha = \beta + \delta_1 + \cdots + \delta_p$  is a chain.*

We can now describe the coefficient functions  $a_{\gamma,k}^{\alpha,j}$ .

LEMMA 12. *For every root  $\alpha \in \Sigma_+$  and  $j = 1, \dots, m_\alpha$  we have*

$$(30) \quad a_{\gamma,k}^{\alpha,j} = \begin{cases} 0 & \text{if } \text{ht}(\alpha) \geq \text{ht}(\gamma) \text{ and } \alpha \neq \gamma \\ 0 & \text{if } \alpha = \gamma \text{ and } k \neq j \\ 1 & \text{if } \alpha = \gamma \text{ and } k = j \\ P & \text{if } \text{ht}(\alpha) < \text{ht}(\gamma), \end{cases}$$

where  $P$  is a polynomial that does not vanish only if  $\alpha \preceq \gamma$ . In this case, it depends only on those variables labeled by those roots  $\alpha_1, \dots, \alpha_q$  for which  $\alpha + \alpha_1 + \cdots + \alpha_q = \gamma$  is a chain. This implies that

$$(31) \quad X_{\alpha,j} = \sum_{\gamma \in \mathcal{C}} \sum_{k=1}^{m_\gamma} a_{\gamma,k}^{\alpha,j} \frac{\partial}{\partial x_{\gamma,k}},$$

for every  $\alpha \in \mathcal{C}$ .

PROOF. Consider  $\{E_{\beta,i} : \beta \in \Sigma_+, 1 \leq i \leq m_\beta\}$  as a basis of  $\mathfrak{n}$  viewed as the tangent space to  $N$  at the identity and write

$$n = \exp \left( \sum_{\epsilon \in \Sigma_+} \sum_{s=1}^{m_\epsilon} y_{\epsilon,s} E_{\epsilon,s} \right) \quad n' = \exp \left( \sum_{\beta \in \Sigma_+} \sum_{r=1}^{m_\beta} x_{\beta,r} E_{\beta,r} \right).$$

Let  $f$  be a smooth function on  $N$ . From the left invariance

$$(X_{\alpha,j} f)(n) = (l_n)_{*e} \frac{\partial}{\partial x_{\alpha,j}} \Big|_e f = \frac{\partial}{\partial x_{\alpha,j}} f \circ l_n(e)$$

it follows that the component  $a_{\gamma,k}^{\alpha,j}$  is given as the derivative with respect to  $x_{\alpha,j}$  of  $f \circ l_n$ , where  $f$  is the coordinate function  $n' \mapsto (n')_{(\gamma,k)}$ . That is,

$$a_{\gamma,k}^{\alpha,j}(n') = \frac{\partial}{\partial x_{\alpha,j}} (nn')_{(\gamma,k)}.$$

An explicit calculation using the Campbell-Hausdorff formula gives

$$\begin{aligned}
 nn' &= \exp \left( \sum_{\epsilon \in \Sigma_+} \sum_{s=1}^{m_\epsilon} y_{\epsilon,s} E_{\epsilon,s} \right) \exp \left( \sum_{\beta \in \Sigma_+} \sum_{r=1}^{m_\beta} x_{\beta,r} E_{\beta,r} \right) \\
 &= \exp \left( \sum_{\epsilon \in \Sigma_+} \sum_{s=1}^{m_\epsilon} y_{\epsilon,s} E_{\epsilon,s} + \sum_{\beta \in \Sigma_+} \sum_{r=1}^{m_\beta} x_{\beta,r} E_{\beta,r} + \right. \\
 &\quad \left. + \frac{1}{2} \sum_{\epsilon \in \Sigma_+} \sum_{s=1}^{m_\epsilon} \sum_{\beta \in \Sigma_+} \sum_{r=1}^{m_\beta} y_{\epsilon,s} x_{\beta,r} [E_{\epsilon,s}, E_{\beta,r}] + \dots \right). \tag{32}
 \end{aligned}$$

Since  $\mathfrak{n}$  is nilpotent, the sum in (32) is finite, and the variable  $x_{\alpha,j}$  appears in the coefficient of an iterated bracket of the form

$$[E_{\epsilon_i,s}, [\dots, [E_{\epsilon_1,s}, [E_{\beta_k,r}, \dots, [E_{\beta_1,r}, E_{\alpha,j}]] \dots]],$$

provided that the bracket is not zero. If it appears with a power greater than or equal to two, then the derivative of the corresponding monomial with respect to  $x_{\alpha,j}$  evaluated at the identity is zero. Thus, the only relevant brackets are those of the form

$$[E_{\epsilon_i,s}, [\dots, [E_{\epsilon_1,s}, E_{\alpha,j}]]],$$

where  $\alpha + \epsilon_1 + \dots + \epsilon_i = \gamma$  is a chain. The coefficient of such an iterated bracket is the monomial  $y_{\epsilon_i,s} \dots y_{\epsilon_1,s} x_{\alpha,j}$ . Its derivative with respect to  $x_{\alpha,j}$  evaluated at the identity is  $y_{\epsilon_i,s} \dots y_{\epsilon_1,s}$ . This proves (30).

Finally, by definition of  $\mathcal{R}$ , if  $\alpha \in \mathcal{C}$  then  $\alpha + \delta \in \mathcal{C}$ , for any simple root  $\delta$  such that  $\alpha + \delta$  is a root. Hence, if  $\alpha \in \mathcal{C}$ , there are no chains going from  $\alpha$  to a root  $\gamma \in \mathcal{R}$ . Thus also (31) follows.  $\square$

For every  $\alpha \in \Sigma_+$ , and  $1 \leq j \leq m_\alpha$ , consider the vector field  $\bar{X}_{\alpha,j}$  whose  $(\gamma, k)$  component is

$$r_{\gamma,k}^{\alpha,j} = \begin{cases} a_{\gamma,k}^{\alpha,j} & \text{if } \gamma \in \mathcal{R} \text{ and } k = 1, \dots, m_\gamma \\ 0 & \text{otherwise.} \end{cases}$$

The  $\bar{X}_{\alpha,j}$  are vector fields on  $S$ , and from (31)  $\bar{X}_{\alpha,j} = 0$  for every  $\alpha \in \mathcal{C}$ . Moreover, (30) implies that the set  $\{\bar{X}_{\alpha,j} : \alpha \in \mathcal{R}, j = 1, \dots, m_\alpha\}$  is a basis of the tangent space at any point of  $S$ . Indeed, writing the matrix of the coefficients of  $\{\bar{X}_{\alpha,j} : \alpha \in \mathcal{R}, j = 1, \dots, m_\alpha\}$ , ordering the roots according to any lexicographic order, we obtain a triangular matrix with ones along

the diagonal. Hence  $\{\phi_*(\bar{X}_{\alpha,j}) : \alpha \in \mathcal{R}, j = 1, \dots, m_\alpha\}$  is a basis of the tangent space at all points of an open set of  $\text{Hess}_{\mathcal{R}}(H)$ .

Next, we introduce some special sub-bundles of the tangent bundle of  $S$ . For  $\delta \in \Delta \cap \mathcal{R}$ , put  $\bar{\mathfrak{g}}_\delta = \text{span}\{\bar{X}_{\delta,i} : i = 1, \dots, m_\delta\}$ . From Proposition 13 below it follows that the vector fields in the family  $\{\bar{\mathfrak{g}}_\delta\}_{\delta \in \Delta_{\mathcal{R}}}$ ,  $\Delta_{\mathcal{R}} = \Delta \cap \mathcal{R}$ , satisfy a Hörmander-type condition: their iterated brackets generate at each point the tangent space of  $S$ . We denote by  $\mathfrak{X}(N)$  the Lie algebra of all smooth vector fields on  $N$ .

**PROPOSITION 13.** *Given  $X$  and  $Y \in \mathfrak{X}(N)$ , the following formula holds at every point  $n \in S$*

$$[\bar{X}, \bar{Y}](n) = \overline{[X, Y]}(n).$$

**PROOF.** Let  $X$  and  $Y \in \mathfrak{X}(N)$  and write  $X = \bar{X} + \underline{X}$ , where

$$\bar{X} := \sum_{\beta \in \mathcal{R}} \sum_{i=1}^{m_\beta} r_{\beta,i} \frac{\partial}{\partial x_{\beta,i}}, \quad \underline{X} := \sum_{\gamma \in \mathcal{C}} \sum_{k=1}^{m_\gamma} c_{\gamma,k} \frac{\partial}{\partial x_{\gamma,k}},$$

and similarly  $Y = \bar{Y} + \underline{Y}$ . Then

$$\overline{[X, Y]}(n) = \overline{[\bar{X}, \bar{Y}]}(n) + \overline{[\bar{X}, \underline{Y}]}(n) + \overline{[\underline{X}, \bar{Y}]}(n) + \overline{[\underline{X}, \underline{Y}]}(n).$$

Clearly  $\overline{[\bar{X}, \bar{Y}]} = [\bar{X}, \bar{Y}]$ . Moreover

$$\overline{[\bar{X}, \underline{Y}]} = \overline{[\underline{X}, \bar{Y}]} = 0,$$

because when expanded in terms of partial derivatives, each of the above brackets contains only coefficients of the form  $(\partial/\partial x_{\gamma,k})r_{\beta,i}$ , which vanish whenever  $\gamma \in \mathcal{C}$  and  $\beta \in \mathcal{R}$  because of (30). Finally,  $\overline{[\underline{X}, \underline{Y}]} = 0$ , because in  $[\underline{X}, \underline{Y}]$  only the coefficients of components labeled by  $\mathcal{C}$  will appear, but they become zero once we project them on  $S$ .  $\square$

Let  $\mathcal{A}, \mathcal{B}$  be some open subsets of  $\text{Hess}_{\mathcal{R}}(H)$ . Without loss of generality, we can assume  $\mathcal{A}, \mathcal{B} \subset (N \cap \text{Hess}_{\mathcal{R}}(H))$ . Let  $f : \mathcal{A} \rightarrow \mathcal{B}$  be a diffeomorphism. We say that  $f$  is a *multicontact* map if

$$f_*(\phi_*(\bar{\mathfrak{g}}_\delta)) \subseteq \phi_*(\bar{\mathfrak{g}}_\delta)$$

for every simple root  $\delta$  in  $\mathcal{R}$ , where  $\phi$  is the graph mapping defined in (29).

**1.2. Example.** Consider the simple Lie algebra  $\mathfrak{g} = \mathfrak{sl}(4, \mathbb{R})$ . The Cartan subspace  $\mathfrak{a}$  is the abelian algebra of diagonal matrices. Denote by  $\text{diag}(a, b, c, d)$  the diagonal matrix with entries  $a, b, c, d$ . The standard simple restricted roots are  $\alpha, \beta, \gamma$ , where  $\alpha(\text{diag}(a, b, c, d)) = (a - b)$ ,  $\beta(\text{diag}(a, b, c, d)) = (b - c)$  and  $\gamma(\text{diag}(a, b, c, d)) = c - d$ . Let  $\mathcal{R}$  be the set of all positive roots except the highest one, that is  $\alpha + \beta + \gamma$ . A natural regular element in  $\mathfrak{a}$  is

$$H = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Consider then  $G = \text{SL}(4, \mathbb{R})$  and its minimal parabolic subgroup consisting of lower triangular matrices. Since the Hessenberg manifold is a submanifold of  $G/P$  and the problem of multicontact mappings is local, we restrict ourselves to the big cell  $N \subset G/P$  and fix coordinates on it:

$$(33) \quad n = \begin{bmatrix} 1 & x & u & z \\ 0 & 1 & y & v \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus

$$\text{Ad } n^{-1} H = \begin{bmatrix} -1 & 3/2x & (u - 3xy)/2 & 2z - (3vx - ut + 3xyt)/2 \\ 0 & 1/2 & -y & (v + yt)/2 \\ 0 & 0 & -1/2 & 3/2t \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The points  $n(x, y, t, u, v, z) \in N$  that lie in the Hessenberg manifold are those that satisfy  $2z - (3vx - ut + 3xyt)/2 = 0$ . Therefore, the slice  $S$  is the algebraic submanifold of  $N$  defined by the equation  $z = 0$ . A basis of  $\mathfrak{n}$  is given by the following left invariant vector fields

$$\begin{aligned} X &= \frac{\partial}{\partial x} + y \frac{\partial}{\partial u} + v \frac{\partial}{\partial z} & U &= \frac{\partial}{\partial u} + t \frac{\partial}{\partial z} & Z &= \frac{\partial}{\partial z}, \\ Y &= \frac{\partial}{\partial y} + t \frac{\partial}{\partial v} & V &= \frac{\partial}{\partial v}, \\ T &= \frac{\partial}{\partial t}, \end{aligned}$$

with nonzero brackets  $[X, Y] = -U$ ,  $[Y, T] = -V$ ,  $[U, T] = -Z$  and  $[X, V] = -Z$ . By projecting  $\partial/\partial z$  to zero, we restrict the above vector

fields to a pointwise basis for the tangent space to  $S$ :

$$\begin{aligned}\overline{X} &= \frac{\partial}{\partial x} + y \frac{\partial}{\partial u}, & \overline{U} &= \frac{\partial}{\partial u}, \\ \overline{Y} &= \frac{\partial}{\partial y} + t \frac{\partial}{\partial v}, & \overline{V} &= \frac{\partial}{\partial v}, \\ \overline{T} &= \frac{\partial}{\partial t}.\end{aligned}$$

The nonzero brackets are  $[\overline{X}, \overline{Y}] = -\overline{U}$  and  $[\overline{Y}, \overline{T}] = -\overline{V}$ . A diffeomorphism  $f$  on some open set in  $S$  is a multicontact mapping if its differential  $f_*$  preserves each of the following subspaces

$$\bar{\mathfrak{g}}_\alpha = \text{span}\{\overline{X}\} \quad \bar{\mathfrak{g}}_\beta = \text{span}\{\overline{Y}\} \quad \bar{\mathfrak{g}}_\gamma = \text{span}\{\overline{T}\}.$$

If we know the multicontact mappings on  $S$  we also know those on the corresponding Hessenberg manifold, because of the diffeomorphism  $\phi$ . For completeness, we compute the stratified structure in the tangent space of the Hessenberg manifold. Since

$$\phi(x, y, t, u, v, 0) = (x, y, t, u, v, (3vx - ut + 3xyt)/4),$$

for a point  $p = \phi^{-1}(q)$  in  $S$  we have

$$\begin{aligned}(\phi_*(\overline{X}))g(q) &= \overline{X}(g \circ \phi)(p) \\ &= \overline{X}g\left(x, y, t, u, v, \frac{(3vx - ut + 3xyt)}{4}\right) \\ &= \frac{\partial}{\partial x}g(q) + y \frac{\partial}{\partial u}g(q) + \frac{3v + 2yt}{4} \frac{\partial}{\partial z}g(q),\end{aligned}$$

for any smooth function  $g$  on  $\text{Hess}_R(H)$ . Thus

$$\phi_*(\overline{X}) = \frac{\partial}{\partial x} + y \frac{\partial}{\partial u} + \frac{3v - 4yt}{4} \frac{\partial}{\partial z}$$

and similarly

$$\begin{aligned}\phi_*(\overline{Y}) &= \frac{\partial}{\partial y} + t \frac{\partial}{\partial v}, \\ \phi_*(\overline{T}) &= \frac{\partial}{\partial t} + \frac{(3xy - u)}{4} \frac{\partial}{\partial z}.\end{aligned}$$

In [11] and [12], the authors study the group of multicontact mappings on the boundaries  $G/P$ . Their approach is to lift the problem to the Lie algebra level. In fact, most of the effort consists in investigating the set of multicontact vector fields, that is vector fields whose local flow is given

by multicontact mappings. The set of multicontact vector fields is a Lie algebra with respect to the Lie bracket. In particular, they ask whether this algebra is finite dimensional and which is the resulting algebra. The integration step to the group level is then relatively simple, and it uses classical tools of Lie theory.

In the case we are studying, we use the same approach and we ask ourselves if the Lie algebra of multicontact vector fields on  $S$  is finite dimensional and how we can characterize this algebra. As we shall see in the next section, the multicontact condition for a vector field  $F$  is equivalent to the fact that  $\text{ad}F$  preserves each  $\bar{\mathfrak{g}}_\delta$ ,  $\delta \in \Delta_{\mathcal{R}}$ . In the present example this means

$$[F, \bar{X}] = \lambda \bar{X} \quad [F, \bar{Y}] = \mu \bar{Y} \quad [F, \bar{T}] = \nu \bar{T},$$

for some smooth functions  $\lambda$ ,  $\mu$  and  $\nu$  on  $S$ . Put  $F = f_x \bar{X} + f_y \bar{Y} + f_t \bar{T} + f_u \bar{U} + f_v \bar{V}$ . The equation  $[F, \bar{X}] = \lambda \bar{X}$  gives

$$\begin{aligned} [f_x \bar{X} + f_y \bar{Y} + f_t \bar{T} + f_u \bar{U} + f_v \bar{V}, \bar{X}] &= -\bar{X}(f_x) \bar{X} + f_y \bar{U} - \bar{X}(f_y) \bar{Y} \\ &\quad - \bar{X}(f_t) \bar{T} - \bar{X}(f_u) \bar{U} - \bar{X}(f_v) \bar{V} \\ &= \lambda \bar{X}, \end{aligned}$$

whence

$$\begin{cases} \bar{X}(f_x) = -\lambda \\ \bar{X}(f_y) = \bar{X}(f_t) = \bar{X}(f_v) = 0 \\ \bar{X}(f_u) = f_y. \end{cases}$$

Similarly,  $[F, \bar{Y}] = \mu \bar{Y}$  yields

$$\begin{cases} \bar{Y}(f_y) = -\mu \\ \bar{Y}(f_x) = \bar{Y}(f_t) = 0 \\ \bar{Y}(f_u) = -f_x \\ \bar{Y}(f_v) = f_t \end{cases}$$

and  $[F, \bar{T}] = \nu \bar{T}$  implies

$$\begin{cases} \bar{T}(f_t) = -\nu \\ \bar{T}(f_x) = \bar{T}(f_y) = \bar{T}(f_u) = 0 \\ \bar{T}(f_v) = -f_y. \end{cases}$$

The equations  $\bar{X}(f_u) = f_y$ ,  $\bar{Y}(f_u) = -f_x$  and  $\bar{Y}(f_v) = f_t$  show that the coefficients  $f_u$  and  $f_v$  determine all the others. The equations involving  $f_u$

alone may be viewed as the pair of systems

$$(34) \quad \begin{cases} \bar{X}^2 f_u = 0 \\ \bar{Y}^2 f_u = 0 \end{cases} \quad \begin{cases} \bar{T} f_u = 0 \\ \bar{TY} f_u = 0, \end{cases}$$

whereas for  $f_v$  we have

$$(35) \quad \begin{cases} \bar{Y}^2 f_v = 0 \\ \bar{T}^2 f_v = 0 \end{cases} \quad \begin{cases} \bar{X} f_v = 0 \\ \bar{XY} f_v = 0. \end{cases}$$

Finally,  $f_u$  and  $f_v$  are linked by the extra cross-condition

$$(36) \quad \bar{X} f_u = \bar{T} f_v.$$

The set of equations above are typical of the problem of multicontact vector fields, and they are related to the multicontact-type equations in the case G/P studied in [11] and [12], in a sense that we show below.

Look first at the systems (34). The second system shows that  $f_u$  is independent of the variables  $v$  and  $t$ . Indeed,

$$\bar{T} f_u = \frac{\partial}{\partial t} f_u = 0$$

implies  $f_u = f_u(x, y, u, v)$ , and

$$\bar{TY} f_u = \frac{\partial}{\partial t} \frac{\partial}{\partial y} f_u + \frac{\partial}{\partial v} f_u + t \frac{\partial}{\partial t} \frac{\partial}{\partial v} f_u = \frac{\partial}{\partial v} f_u = 0$$

implies  $f_u = f_u(x, y, u)$ . The systems (34) are then equivalent to the system (8) of Chap. 2 when investigating multicontact vector fields on the nilpotent Iwasawa subgroup of  $\mathrm{SL}(3, \mathbb{R})$ . It can be integrated and yields the following explicit polynomial solution

$$f_u = a_0 + a_1 x + a_2 y + a_3 u + a_4 x y + a_5 x(u - x y) + a_6 u y + a_7 u(u - x y).$$

The same argument holds for (35) and one obtains

$$f_v = b_0 + b_1 y + b_2 t + b_3 v + b_4 y t + b_5 y(v - y t) + b_6 t v + b_7 v(v - y t).$$

Thus, we may view the study of multicontact vector fields on the slice

$$S = \left\{ \begin{bmatrix} 1 & x & u & 0 \\ 0 & 1 & y & v \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix} : x, y, t, u, v \in \mathbb{R} \right\}$$

as the study of two embedded models corresponding to  $\mathrm{SL}(3, \mathbb{R})$  that must satisfy the additional compatibility condition given by (36). Using this last condition, the polynomials  $f_v$  and  $f_u$  become

$$(37) \quad f_v = b_0 + b_1y + b_2t + b_3v + b_4yt + b_5y(v - yt)$$

$$(38) \quad f_u = a_0 - b_2x + a_2y - (a_4 + b_4)u + a_4xy + b_5uy.$$

The Lie algebra of multicontact vector fields has dimension nine, that is the number of free constants appearing in the expressions of  $f_u$  and  $f_v$  because to each possible pair  $(f_u, f_v)$  there corresponds one and only one multicontact vector field. This answers to the question of finite dimensionality.

We now want to characterize the Lie algebra of multicontact vector fields. In order to identify this Lie algebra, we refer again to [11]. Indeed, the problem we are considering is nothing but the study of multicontact mappings on  $N$  projected to the slice that corresponds to setting one coordinate equal to zero. Recall that the special subbundles defining the multicontact structure are obtained by projecting onto  $S$  those vector fields on  $N$  that correspond to simple roots, and that a multicontact mapping is by definition a map that preserves these bundles. Thus, we consider the multicontact vector fields on  $N$ , project them to be tangent to  $S$  and finally check whether we obtain multicontact vector fields on  $S$ .

By [11], the multicontact vector fields on  $N$  are all of the form  $\tau(X)$  for some  $X \in \mathfrak{g}$ , where

$$\tau(X)f(n) = \frac{d}{ds}f(\exp(-sX)n)\Big|_{s=0},$$

and  $n \mapsto \exp(-sX)n$  is the action of  $\exp(-sX) \in G$  on  $G/P$ , restricted to  $N$ , namely  $\exp(-sX)n$  is the  $N$ -component of the product in the Bruhat decomposition of  $G/P$ .

Consider a vector generating the root space  $\mathfrak{g}_\alpha$ , namely  $E_{1,2}$ , where  $E_{i,j}$  denotes the matrix in  $\mathfrak{sl}(4, \mathbb{R})$  with 1 in the  $(i, j)$  position and zero elsewhere. Therefore, using the coordinates introduced in (33)

$$\exp(-sE_{1,2})n = \begin{bmatrix} 1 & -s & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} n = \begin{bmatrix} 1 & x-s & u & z \\ 0 & 1 & y & v \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

whence

$$\tau(E_{1,2})f(n) = -\frac{\partial}{\partial x}f(n) = -Xf(n) + yUf(n) + (v - yt)Zf(n).$$

The projected vector field obtained by setting  $\partial/\partial z = 0$ , namely

$$\overline{\tau(E_{1,2})} = -\overline{X} + y\overline{U},$$

is tangent to  $S$  at each point by construction. Moreover, the components  $(f_u, f_v) = (y, 0)$  satisfy equations (34), (35) and (36), so that  $\overline{\tau(E_{1,2})}$  is a multicontact vector field on  $S$ . If we do the same calculation for each element in the canonical basis of  $\mathfrak{sl}(4, \mathbb{R})$ , we obtain polynomials of the form (37) and (38) for all matrices of the form

$$\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{bmatrix}.$$

In particular we see that  $\overline{\tau(E_{1,4})} = 0$ , and no other matrix in  $\mathfrak{sl}(4, \mathbb{R})$  of the form described above gives a zero vector field. Therefore the Lie algebra of multicontact vector fields on  $S$  is isomorphic to  $\mathfrak{q}/\mathfrak{n}_C$ , where

$$\mathfrak{q} = \text{span} \left\{ \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{bmatrix} : * \in \mathbb{R} \right\} \cap \mathfrak{sl}(4, \mathbb{R})$$

and

$$\mathfrak{n}_C = \text{span} \left\{ \begin{bmatrix} 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} : * \in \mathbb{R} \right\}.$$

The Lie algebra  $\mathfrak{q}/\mathfrak{n}_C$  can be viewed as the union of the two submodels of  $\mathfrak{sl}$ -type plus the reflection of their intersection. In the subsequent sections we study the differential equations defining the multicontact vector fields on an arbitrary Hessenberg manifold. On the one hand we shall show that  $\mathfrak{q}/\mathfrak{n}_C$  always defines a set of multicontact vector fields. On the other hand, we shall see that the converse of this statement is not true in general, that is the Lie algebra of multicontact vector fields can be bigger than  $\mathfrak{q}/\mathfrak{n}_C$ . It does, however, become true under the additional hypothesis that the Hessenberg structure encodes a finite number of embedded models (i.e. each corresponding to an Iwasawa nilpotent group  $N$ ) that intersect non trivially. This is exactly what happens in the case study that we have just discussed.

## 2. A set of multicontact vector fields

In this section we consider the infinitesimal version of the notion of multicontact mapping. We obtain a Lie algebra of multicontact vector fields and we address the problem of understanding its structure.

**2.1. Lifting the multicontact conditions to the infinitesimal level.** Since the composition and the inverse of multicontact maps on a Hessenberg manifold are multicontact maps, the set of such maps is a group. Moreover, as we already noticed, all Hessenberg manifolds corresponding to different choices of regular  $H$  give rise to the same slice  $S$ , so they are mutually diffeomorphic. Thus the group of multicontact maps does not depend on  $H$ , so that from now on we focus our attention on the slice  $S$  of  $N$ . Fix an open set  $\mathcal{A}$  of  $S$ . In order to characterize the group of multicontact maps, we lift the problem to the Lie algebra level, by considering multicontact vector fields, that is, vector fields  $F$  on  $\mathcal{A}$  whose local flow  $\{\psi_t^F\}$  consists of multicontact maps. This means that if  $\delta \in \Delta_{\mathcal{R}}$ , then

$$\frac{d}{dt}(\psi_t^F)_*(\overline{X}_\delta) \Big|_{t=0} = -\mathcal{L}_F(\overline{X}_\delta) = [\overline{X}_\delta, F],$$

where  $\mathcal{L}$  denotes the Lie derivative. Hence a smooth vector field  $F$  on  $\mathcal{A}$  is a multicontact vector field if and only if

$$(39) \quad [F, \overline{\mathfrak{g}}_\delta] \subseteq \overline{\mathfrak{g}}_\delta \text{ for every } \delta \in \Delta_{\mathcal{R}}.$$

We write a vector field on  $\mathcal{A}$  as

$$(40) \quad F = \sum_{\gamma \in \mathcal{R}} \sum_{j=1}^{m_\gamma} f_{\gamma,j} \overline{X}_{\gamma,j},$$

where  $f_{\gamma,j}$  are smooth functions on  $\mathcal{A}$ . Condition (39) becomes

$$[F, \overline{X}_{\delta,i}] = \sum_{k=1}^{m_\delta} \lambda_{\delta,k}^i \overline{X}_{\delta,k}, \quad \delta \in \Delta_{\mathcal{R}}, i = 1, \dots, m_\delta,$$

where  $\{\lambda_{\delta,k}^i\}$  is a set of smooth functions. By Proposition 13 we have

$$\begin{aligned} [\overline{X}_{\alpha,i}, \overline{X}_{\beta,j}] &= \overline{[X_{\alpha,i}, X_{\beta,j}]} \\ &= \sum_{k=1}^{m_{\alpha+\beta}} c_{\alpha,\beta}^{ijk} \overline{X}_{\alpha+\beta,k} \quad \alpha, \beta \in \mathcal{R}, \alpha + \beta \in \Sigma, \end{aligned}$$

where  $c_{\alpha,\beta}^{ijk}$  are the structure constants with respect to the chosen basis, and  $\overline{X}_{\alpha+\beta} = 0$  if  $\alpha + \beta \in \mathcal{C}$ . We can write the multicontact conditions as the system of equations

$$\sum_{\gamma \in \mathcal{R}} \sum_{j=1}^{m_\gamma} \overline{X}_{\delta,i}(f_{\gamma,j}) \overline{X}_{\gamma,j} + \sum_{\gamma \in \mathcal{R}} \sum_{j=1}^{m_\gamma} \left( \sum_{l=1}^{m_{\gamma-\delta}} c_{\delta,\gamma-\delta}^{ilj} f_{\gamma-\delta,l} \right) \overline{X}_{\gamma,j} = - \sum_{j=1}^{m_\delta} \lambda_{\delta,j}^i \overline{X}_{\delta,j},$$

as  $\delta$  varies in  $\Delta_{\mathcal{R}}$  and  $i = 1, \dots, m_\delta$ . Equivalently,  $F$  is a multicontact vector field on  $\mathcal{A}$  if and only if for all  $\gamma \in \mathcal{R}$  and some functions  $\{\lambda_{\delta,j}^i\}$  the following equations are satisfied on  $\mathcal{A}$ :

$$(41) \quad \begin{cases} \overline{X}_{\delta,i}(f_{\delta,j}) = -\lambda_{\delta,i}^j \\ \overline{X}_{\delta,i}(f_{\gamma,j}) = 0 & \text{if } \gamma - \delta \notin \Sigma_+ \cup \{0\} \\ \overline{X}_{\delta,i}(f_{\gamma,j}) + \sum_{l=1}^{m_{\gamma-\delta}} c_{\delta,\gamma-\delta}^{ilj} f_{\gamma-\delta,l} = 0 & \text{if } \gamma - \delta \in \Sigma_+ \end{cases}$$

for all the simple roots  $\delta$  in  $\Delta_{\mathcal{R}}$  and  $1 \leq i, j \leq m_\delta$ . We may clearly forget the equation  $\overline{X}_{\delta,i}(f_{\delta,j}) = -\lambda_{\delta,i}^j$  because  $\lambda_{\delta,i}^j$  is arbitrary.

We fix some notations. We write  $MC(N)$  and  $MC(S)$  for the Lie algebra of multicontact vector fields on some open subset of  $N$  and  $S$  respectively. Let  $\mathcal{C}$  be the complement in  $\Sigma_+$  of some Hessenberg type set. We say that a function  $f$  on  $N$  is  $\mathcal{C}$ -independent if it does not depend on the coordinates labeled by  $\mathcal{C}$ . We record a simple consequence of (41) for later reference.

**LEMMA 14.** *Let  $F \in MC(S)$  be as in (40). Then  $f_{\gamma,j}$  are  $\mathcal{C}$ -independent for every  $\gamma \in \mathcal{R}$ .*

**PROOF.** All the  $f_{\gamma,j}$  appearing in (40) are functions on  $S$ .  $\square$

Because of Lemma 14, if  $F \in MC(S)$  is as in (40), then  $\overline{X}_{\delta,i} f_{\gamma,j} = X_{\delta,i} f_{\gamma,j}$ . Thus, from now on we shall write  $X_{\delta,i}$  in place of  $\overline{X}_{\delta,i}$  whenever treating multicontact vector fields, if no ambiguity arises.

From (31) of Lemma 12 it follows that if  $\mathcal{R}$  is a Hessenberg type set of roots and  $\gamma \in \mathcal{C}$ , then a (basis) left invariant vector field  $X_{\gamma,k}$  on  $N$  does not depend on the partial derivative vector fields that are labeled by the positive roots in  $\mathcal{C}$ . This implies in particular that the system of equations

$$(42) \quad X_{\gamma,k} f = 0 \quad \text{for every } \gamma \in \mathcal{C} \text{ and } k = 1, \dots, m_\gamma$$

is equivalent to the  $\mathcal{C}$ -independence, namely to

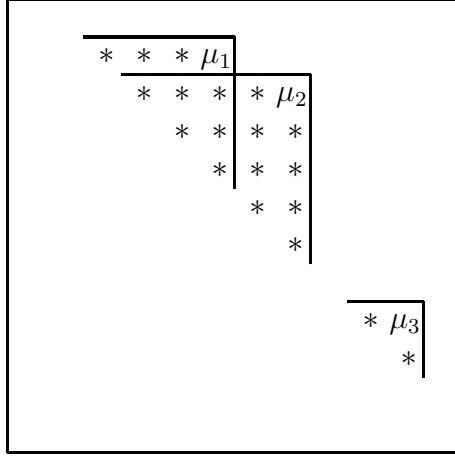
$$(43) \quad \frac{\partial}{\partial x_{\gamma,k}} f = 0 \quad \text{for every } \gamma \in \mathcal{C} \text{ and } k = 1, \dots, m_\gamma.$$

**2.2. Dark zones.** We split (41) into suitable independent subsystems, each defining multicontact vector fields on some Hessenberg manifold of lower dimension, and we show that we can focus our attention to only one of them at a time.

Call a positive root  $\mu$  in  $\mathcal{R}$  *maximal* if  $\mu + \alpha \notin \mathcal{R}$  for any other root  $\alpha \in \Sigma_+$ . Since, by definition of  $\mathcal{R}$ ,  $\mu + \alpha \notin \mathcal{R}$  if  $\alpha \in \mathcal{C}$ , it suffices to check maximality for all  $\alpha \in \mathcal{R}$ . Denote by  $\mathcal{R}_M$  the set of maximal roots. For a fixed  $\mu \in \mathcal{R}_M$ , we call *shadow* of  $\mu$  the set

$$S_\mu = \{\alpha \in \mathcal{R} : \alpha \preceq \mu\}.$$

Notice that  $S_\mu \neq \emptyset$  because  $\mu \in S_\mu$ . The union  $\bigcup_{\mu \in \mathcal{R}_M} S_\mu$  covers  $\mathcal{R}$ . Indeed, let  $\alpha \in \mathcal{R}$ . Either  $\alpha \in \mathcal{R}_M$ , or there exists  $\beta \in \mathcal{R}$  such that  $\alpha + \beta \in \mathcal{R}$ . Again,  $\alpha + \beta$  can be maximal or not. If it is maximal, then  $\alpha$  lies in its shadow. If it is not maximal, then we may continue this process and in a finite number of steps we reach a maximal root in whose shadow  $\alpha$  lies.



In the picture above we see a representation of a Hessenberg set  $\mathcal{R}$  relative to  $\mathfrak{sl}(n, \mathbb{R})$  consisting of three shadows. Here  $\mathcal{R}_M = \{\mu_1, \mu_2, \mu_3\}$ . For simplicity, we label by  $\mu_i$  the matrix entry that corresponds to the root space  $\mathfrak{g}_{\mu_i}$ . The direct sum of all the root spaces associated to roots in the shadow  $\mathcal{S}_{\mu_i}$  is a subspace whose coordinates belong to the conical sector

that has  $\mathbf{g}_{\mu_i}$  as north-east corner. If one sees these corners as idealized obstacles to a light beam coming from a far north-east point, then each cone is the shadow produced by it. This explains the terminology.

We partition  $\mathcal{R}$  into the disjoint union of *dark zones*, a dark zone being a connected component of  $\mathcal{R}$  in a loose sense, that is, a maximal union of shadows  $\mathcal{Z} = \cup_{i=1}^k \mathcal{S}_{\mu_i}$  with the property that either  $k = 1$  or any  $\mathcal{S}_{\mu_i}$  intersects at least another  $\mathcal{S}_{\mu_j}$  in the same dark zone. In the picture above,  $\mathcal{R}$  is the union of two dark zones: one is the union  $\mathcal{S}_{\mu_1} \cup \mathcal{S}_{\mu_2}$  and the other consists of the single shadow  $\mathcal{S}_{\mu_3}$ .

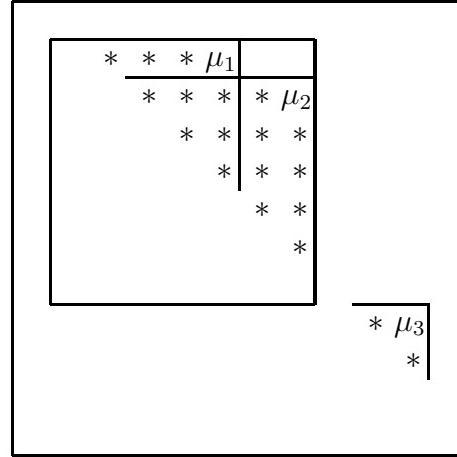
By their very definition, dark zones are disjoint. This will allow us to reduce the problem of solving (41) to the problem of solving several simpler systems, each naturally associated to a dark zone. Suppose that  $\mathcal{Z}_1, \dots, \mathcal{Z}_p$  is a numbering of the dark zones of  $\mathcal{R}$ . Given  $F \in \mathfrak{X}(S)$  as in (40), we write

$$F = \sum_{i=1}^p F_i,$$

where

$$F_i = \sum_{\gamma \in \mathcal{Z}_i} \sum_{j=1}^{m_\gamma} f_{\gamma,j} \overline{X}_{\gamma,j}.$$

Clearly, each  $F_i$  is itself a vector field in  $\mathfrak{X}(S)$ . Since  $F_i$  picks the components of  $F$  along the directions labeled by  $\mathcal{Z}_i$ , it is natural to consider the sub-slice of  $S$  that corresponds to it, as we now explain.



Fix a dark zone  $\mathcal{Z}$ . The set of roots contained in  $\mathcal{Z}$  generate the positive set of an irreducible root system, say  $\Sigma_+(\mathcal{Z})$ , and the corresponding Lie

algebra

$$\mathfrak{n}(\mathcal{Z}) = \bigoplus_{\beta \in \Sigma_+(\mathcal{Z})} \mathfrak{g}_\beta$$

is a nilpotent Iwasawa algebra. The roots in  $\mathcal{Z}$  play, within  $\Sigma_+(\mathcal{Z})$ , the rôle of a Hessenberg set of roots. Also,  $\mathfrak{n}(\mathcal{Z})$  is a subalgebra of  $\mathfrak{n}$  and we may consider the (connected, simply connected, nilpotent) Lie subgroup  $N(\mathcal{Z})$  of  $N$  whose Lie algebra is  $\mathfrak{n}(\mathcal{Z})$ . Thus, if  $\mathcal{Z}$  is a dark zone we write

$$\mathcal{S}_{\mathcal{Z}} = \{n \in N : x_{\gamma,k} = 0 \text{ if } \gamma \notin \mathcal{Z}\}.$$

Coming back to the decomposition  $\mathcal{R} = \mathcal{Z}_1 \cup \dots \cup \mathcal{Z}_p$ , we write for simplicity  $\mathcal{S}_i$  in place of  $\mathcal{S}_{\mathcal{Z}_i}$ . We want to prove the following reduction result.

**THEOREM 15.** *If  $F \in MC(S)$ , then  $F_i \in MC(S_i)$  for all  $i = 1, \dots, p$ . Conversely, given  $G_i \in MC(S_i)$  with  $i = 1, \dots, p$ , then  $\sum_i G_i \in MC(S)$ .*

The proof requires some remarks, that we state in the next lemmas.

**LEMMA 16.** *Let  $\mathcal{Z} \subset \mathcal{R}$  be a dark zone and let  $\alpha \in \mathcal{Z}$ . The  $(\gamma, k)$  component of the vector field  $X_{\alpha,j}$  is zero for every  $\gamma \in \mathcal{R} \setminus \mathcal{Z}$ .*

**PROOF.** Suppose  $\alpha \in \mathcal{Z}$ ,  $\gamma \in \mathcal{R} \setminus \mathcal{Z}$  and suppose the  $(\gamma, k)$ -component of the vector field  $X_{\alpha,j}$  is not zero. By Lemma 12, there exist roots  $\alpha_1, \dots, \alpha_q$  such that  $\alpha + \alpha_1 + \dots + \alpha_q = \gamma$  is a chain, so that in particular  $\gamma - \alpha_q - \dots - \alpha_j$  is also a root for  $j = 1, \dots, q-1$ . Now, since  $\gamma \in \mathcal{R}$ , then  $\gamma \in S_\mu$  for some maximal root  $\mu$ . Therefore  $\alpha = \gamma - \alpha_q - \dots - \alpha_1 \in S_\mu$ . This implies that both  $\alpha$  and  $\gamma$  belong to the same shadow, and hence to the same dark zone, that is a contradiction.  $\square$

**LEMMA 17.** *The coefficients of a multicontact vector field  $F$  are determined by its  $\mathfrak{g}_\mu$  components, as  $\mu$  varies in  $\mathcal{R}_M$ .*

**PROOF.** The proof of this statement is analogous to the proof of Proposition 3.3 of [12]. We give an outline for the reader's convenience. Let  $\beta$  be a positive root distinct from  $\mu$ . Let  $\delta$  be a simple root such that  $\beta + \delta$  is again a root. The set  $\Delta(\beta)$  of all such simple roots is not empty by Lemma 11. The equations (41) yield

$$X_{\delta,i}(f_{\beta+\delta,j}) + \sum_{l=1}^{m_\beta} c_{\delta,\beta}^{ilj} f_{\beta,l} = 0, \quad \delta \in \Delta(\beta), \quad (i, j) \in \mathcal{I}_\delta(\beta),$$

where  $\mathcal{I}_\delta(\beta) = \{(i, j) : 1 \leq i \leq d_\delta, 1 \leq j \leq d_{\beta+\delta}\}$ ,  $\delta \in \Delta(\beta)$ . We are thus led to consider the linear map given by the matrix  $A = (a_{I,l}) = (c_{\delta,\beta}^{ilj})$  with row index  $I = (i, j)$  and column index  $l$  varying in  $\{1, \dots, d_\beta\}$ . The proof

consists then in showing that  $A$  has rank  $d_\beta$ , because in this case the  $f_{\beta,l}$  are uniquely given by  $X_{\delta,i}(f_{\beta+\delta,j})$ . This may be done using Lemma 18 below, that is proved in [12] as a consequence of Kostant's double transitivity theorem.  $\square$

LEMMA 18. [12] *Let  $\alpha, \beta \in \Sigma$  such that  $\alpha + \beta$  is a root, then*

$$\{[X, Y] : X \in \mathfrak{g}_\alpha, Y \in \mathfrak{g}_\beta\} = \mathfrak{g}_{\alpha+\beta},$$

and  $\{Z \in \mathfrak{g}_\beta : [\mathfrak{g}_\alpha, Z] = \{0\}\} = \{0\}$ .

Lemma 17 suggests a hierachic structure of the equations (41). In particular, if  $\gamma + \delta_1 + \cdots + \delta_s = \alpha$  is a chain, there exist vector fields  $X_1 \in \mathfrak{g}_{\delta_1}, \dots, X_s \in \mathfrak{g}_{\delta_s}$  such that the differential monomial  $X_1 \cdots X_s$  maps a  $\alpha$ -component to a  $\gamma$ -component of a vector field whose coefficients solve (41). In the proof the folowing result, we use again Lemma 18.

LEMMA 19. *Let  $F \in MC(S)$  be as in (40). Then  $X f_{\gamma,j} = 0$  for every  $\gamma \in \mathcal{S}_\mu$ , every  $j = 1, \dots, m_\gamma$  and every  $X \in \mathfrak{g}_\alpha$  with  $\alpha \notin \mathcal{S}_\mu$ .*

PROOF. If  $\alpha \notin \mathcal{S}_\mu$ , then it is either out of  $\mathcal{R}$  or it is in some other shadow. If  $\alpha \in \mathcal{C}$ , then  $X f_{\gamma,j} = 0$  by Lemma 14.

Assume  $\alpha \in \mathcal{R}$ . It is enough to prove the statement for  $\gamma = \mu$ . Indeed, suppose the result true for all  $f_{\mu,j}$ 's. Then, by the equivalence of (42) and (43), these functions are  $(\Sigma_+ \setminus \mathcal{S}_\mu)$ -independent, because  $\mathcal{S}_\mu$  is a Hessenberg type subset. If  $\gamma + \delta_1 + \cdots + \delta_p = \mu$  is a chain, then by Lemma 17 there exist vector fields  $X_1, \dots, X_p$  in  $\mathfrak{g}_{\delta_1}, \dots, \mathfrak{g}_{\delta_p}$  such that  $X_1 \cdots X_p f_{\mu,j} = f_{\gamma,k}$ . Each  $X_i$ ,  $i = 1, \dots, p$ , has the form calculated in Lemma 12, that is

$$X_i = \sum_{\alpha \in \Sigma_+} \sum_{j=1}^{m_\alpha} a_{\alpha,j}^i \frac{\partial}{\partial x_{\alpha,j}},$$

where  $a_{\alpha,j}$  is a nonzero polynomial only if there exists a chain of roots going from  $\delta_i$  to  $\alpha$ . In this case  $a_{\alpha,j}^i$  is a polynomial in the variables  $\{x_{\beta,l}\}$  with  $\beta \prec \alpha$ . In particular this holds for  $i = p$  and we show next that this forces  $X_p f_{\mu,j}$  to be  $(\Sigma_+ \setminus \mathcal{S}_\mu)$ -independent. Indeed, if  $a_{\alpha,j}^p$  depends on some variable in  $(\Sigma_+ \setminus \mathcal{S}_\mu)$ , then  $\alpha \in (\Sigma_+ \setminus \mathcal{S}_\mu)$  and therefore  $\partial f_{\mu,j} / \partial x_{\alpha,k} = 0$  for all  $k = 1, \dots, m_\alpha$ . Hence all coefficients  $f_{\gamma,j}$  with  $\text{ht}(\gamma) = \text{ht}(\mu) - 1$  are  $(\Sigma_+ \setminus \mathcal{S}_\mu)$ -independent. By iterating the same argument, the conclusion holds for every possible height, thus for every  $\gamma$ .

It remains to be proved that the lemma is true for  $f_{\mu,i}$ . If  $\alpha$  is simple, then it is clear by (41) that  $X f_{\mu,j} = 0$ . Let now  $\alpha = \delta_1 + \cdots + \delta_p$  be a non simple root in  $\mathcal{R} \setminus \mathcal{S}_\mu$ . Then there exists  $\delta \in \{\delta_1, \dots, \delta_p\}$  such that

$\delta \notin \mathcal{S}_\mu$ , for otherwise  $\alpha \succ \mu$  and  $\mu$  would not be maximal. By Lemma 18 there exist vector fields  $X_1, \dots, X_p$  in  $\mathfrak{g}_{\delta_1}, \dots, \mathfrak{g}_{\delta_p}$ , respectively, such that  $X = [X_p, [\dots, [X_2, X_1]] \dots]$ . Then there exists a set  $\Lambda$  of permutations of  $p$  elements such that

$$[X_p, [\dots, [X_2, X_1]] \dots] f_{\mu,j} = \left( \sum_{\lambda \in \Lambda} c_\lambda X_{\lambda(1)} \cdot \dots \cdot X_{\lambda(p)} \right) f_{\mu,j},$$

for some constants  $c_\lambda$ . Let  $h \in \{1, \dots, p\}$  be the largest index such that  $\delta_{\lambda(h-1)} \notin \mathcal{S}_\mu$ , so that clearly  $\delta_{\lambda(k)}$  is in  $\mathcal{S}_\mu$  for all  $k \geq h$ . We show that each differential monomial that appears in the sum of the right hand side is zero on  $f_{\mu,j}$ . Consider  $X_{\lambda(i)} \dots X_{\lambda(p)}$ , with  $i \geq h$ . Three possible cases arise.

- (i)  $\mu - \delta_{\lambda(p)} - \dots - \delta_{\lambda(i)} = 0$ , so that  $\mu = \delta_{\lambda(p)} + \dots + \delta_{\lambda(i)}$ . In this case  $\alpha$  is the sum of  $\mu$  and some other simple roots. Hence  $\alpha$  is a root in  $\mathcal{R}$  greater than  $\mu$ , a contradiction.
- (ii) There exists  $i \geq h$  such that  $\mu - \delta_{\lambda(p)} - \dots - \delta_{\lambda(i+1)}$  is a positive root and  $\mu - \delta_{\lambda(p)} - \dots - \delta_{\lambda(i)}$  is not a root. In this case, from Lemma 17 and the remark thereafter, the differential monomial  $X_{\lambda(i+1)} \dots X_{\lambda(p)}$  maps  $f_{\mu,j}$  into a component that belongs to the root space associated to  $\mu - \delta_{\lambda(p)} - \dots - \delta_{\lambda(i+1)}$ , say  $g$ . Since  $\mu - \delta_{\lambda(p)} - \dots - \delta_{\lambda(i+1)} - \delta_{\lambda(i)}$  is not a root,  $X_{\lambda(i)} g = 0$  by (41).
- (iii)  $\mu - \delta_{\lambda(p)} - \dots - \delta_{\lambda(i)}$  is a root for all  $i \geq h$ . Again the differential monomial  $X_{\lambda(h)} \dots X_{\lambda(p)}$  maps  $f_{\mu,j}$  into a component along the root space labeled by  $\mu - \delta_{\lambda(p)} - \dots - \delta_{\lambda(h)}$ . But  $\mu - \delta_{\lambda(p)} - \dots - \delta_{\lambda(h)} - \delta_{\lambda(h-1)}$  is not a root, for otherwise  $\delta_{\lambda(h-1)}$  would lie in  $\mathcal{S}_\mu$ . Therefore we can conclude as in the previous case. Thus  $X_{\lambda(h-1)} \dots X_{\lambda(p)}$  maps the function  $f_{\mu,j}$  to zero.

□

*Proof of Theorem 15.*

“ $\Rightarrow$ ”. Lemma 19 applies in particular to each dark zone, in the sense that a coefficient  $f_{\gamma,k}$  of a multicontact vector field on  $S$  is annihilated by those left invariant vector fields corresponding to the roots that do not belong to the dark zone where  $\gamma$  lies. Since each dark zone plays the rôle of a Hessenberg set of roots, its complement defines an ideal in  $\mathfrak{n}$ , namely

$$\mathfrak{n}_{\mathcal{Z}^c} = \bigoplus_{\alpha \in \Sigma_+ \setminus \mathcal{Z}} \mathfrak{g}_\alpha,$$

where  $\mathcal{Z}^c = \Sigma_+ \setminus \mathcal{Z}$ . The corresponding nilpotent Lie group admits the set  $\{X_{\alpha,j} : \alpha \in \Sigma_+ \setminus \mathcal{Z}\}$  as a basis for its tangent space at each point. From

(31) in Lemma 12, all these vector fields depend on the coordinate vector fields labeled by the positive roots in  $\Sigma_+ \setminus \mathcal{Z}$ . Recall in particular that from (42) and (43)

$$X_{\gamma,k}f = 0 \text{ for all } \gamma \notin \mathcal{Z} \iff \frac{\partial}{\partial x_{\gamma,k}} f = 0 \text{ for all } \gamma \notin \mathcal{Z}.$$

This fact, together with Lemma 19, tells us that the coefficients of the vector field  $F_i$  are functions on  $S_i$ , that is, they are  $(\mathcal{R} \setminus \mathcal{Z})$ -independent.

Moreover, by Lemma 16, the projections  $\overline{X}_\delta$  onto the tangent space at each point of  $S$  are in fact projections on the tangent space of  $S_i$ . Therefore  $F_i \in \mathfrak{X}(S_i)$ . Hence  $F_i$  is in  $MC(S_i)$  if and only if

$$(44) \quad \begin{cases} \overline{X}_{\delta,i}(f_{\gamma,j}) = 0 & \gamma - \delta \notin \Sigma_+ \cup \{0\} \\ \overline{X}_{\delta,i}(f_{\gamma,j}) + \sum_{l=1}^{m_{\gamma-\delta}} c_{\delta,\gamma-\delta}^{ilj} f_{\gamma-\delta,l} = 0 & \gamma - \delta \in \Sigma_+, \end{cases}$$

with  $\delta \in \Delta \cap \mathcal{Z}_i$  and  $\gamma \in \mathcal{Z}_i$ . We conclude by observing that these equations are satisfied by assumption.

“ $\Leftarrow$ ”. Each vector field  $G_i$  can be naturally viewed as a vector field on  $S$ . Furthermore, since each  $G_i$  satisfies the system of equations (44), then the vector field  $\sum_i G_i$  satisfies the system (41). Thus, it defines a multicontact vector field on  $S$ . This concludes the proof of the theorem.  $\square$

Theorem 15 allows us to study each dark zone separately. From now on we thus assume that the Hessenberg set contains all simple roots.

**2.3. A set of solutions.** A further step in investigating the system of differential equations (41) allows us to find a set of solutions.

In [12], the authors determine the multicontact vector fields on the Iwasawa group  $N$ , by solving a system of differential equations similar to (41). In particular, if  $V = \sum_{\gamma \in \Sigma_+} \sum_{j=1}^{m_\gamma} v_{\gamma,j} X_{\gamma,j}$  is a vector field on  $N$ , then  $V$  is of multicontact type if it satisfies the following system of equations

$$(45) \quad \begin{cases} X_{\delta,i}(v_{\gamma,j}) = 0 & \text{if } \gamma - \delta \notin \Sigma_+ \cup \{0\} \\ X_{\delta,i}(v_{\gamma,j}) + \sum_{l=1}^{m_{\gamma-\delta}} c_{\delta,\gamma-\delta}^{ilj} v_{\gamma-\delta,l} = 0 & \text{if } \gamma - \delta \in \Sigma_+, \end{cases}$$

where  $\gamma$  varies in  $\Sigma_+$ ,  $\delta$  in  $\Delta$ , and the  $v_{\gamma,j}$  are smooth functions on  $N$ . Write

$$\bar{V} = \sum_{\gamma \in \Sigma_+} \sum_{j=1}^{m_\gamma} v_{\gamma,j} \bar{X}_{\gamma,j}.$$

If  $V$  solves (45), then the projection  $\bar{V}$  satisfies (41). Moreover, if the coefficients  $v_{\gamma,j}$  are  $\mathcal{C}$ -independent for every  $\gamma \in \mathcal{R}$ , then the vector field  $\bar{V}$  is tangent at each point to  $S$ . Summarizing, in this case  $\bar{V}$  is a multicontact vector field on  $S$ . In [12] it is proved that the multicontact vector fields on  $N$  are all of the form  $\tau(E)$  for some  $E \in \mathfrak{g}$ , where

$$(46) \quad \tau(E)h(n) = \frac{d}{dt} h(\exp(-tE)n) \Big|_{t=0}.$$

We ask ourselves for which  $E \in \mathfrak{g}$  the coefficients of  $\overline{\tau(E)}$  are  $\mathcal{C}$ -independent. Denote by  $\mathfrak{q}$  the parabolic subalgebra of  $\mathfrak{g}$  defined as the normalizer in  $\mathfrak{g}$  of  $\mathfrak{n}_\mathcal{C}$

$$\mathfrak{q} := N_{\mathfrak{g}} \mathfrak{n}_\mathcal{C} = \{X \in \mathfrak{g} : [X, Y] \in \mathfrak{n}_\mathcal{C}, \forall Y \in \mathfrak{n}_\mathcal{C}\}.$$

Clearly  $\mathfrak{q} \supset \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ , so that  $\mathfrak{q}$  is a parabolic subalgebra of  $\mathfrak{g}$ .

**THEOREM 20.** *Let  $\mathcal{R} \subseteq \Sigma_+$  a Hessenberg type set,  $\mathcal{C}$  the complement of  $\mathcal{R}$ , and  $\mathfrak{q} = N_{\mathfrak{g}} \mathfrak{n}_\mathcal{C}$ . For every  $E \in \mathfrak{q}$ ,  $\overline{\tau(E)}$  is a multicontact vector field on  $S$ . In particular, the map*

$$(47) \quad \nu : \mathfrak{q} \longrightarrow \mathfrak{X}(S)$$

*defined by  $\nu(E) = \overline{\tau(E)}$  is a Lie algebra homomorphism. If  $\Delta \subset \mathcal{R}$ , then the kernel of  $\nu$  is  $\mathfrak{n}_\mathcal{C}$ . Thus  $\nu(\mathfrak{q})$  is isomorphic to  $\mathfrak{q}/\mathfrak{n}_\mathcal{C}$ .*

**PROOF.** In order to prove the first claim we show that the coefficients of  $\overline{\tau(E)}$  are  $\mathcal{C}$ -independent for every  $E \in \mathfrak{q}$ . Let  $E' \in \mathfrak{n}_\mathcal{C}$ . Then

$$[\tau(E), \tau(E')] = [\sum_{\alpha \in \mathcal{R}} \sum_{i=1}^{m_\alpha} f_{\alpha,i} X_{\alpha,i} + \sum_{\beta \in \mathcal{C}} \sum_{j=1}^{m_\beta} f_{\beta,j} X_{\beta,j}, \sum_{\gamma \in \mathcal{C}} \sum_{k=1}^{m_\gamma} g_{\gamma,k} X_{\gamma,k}]$$

must lie in  $\tau(\mathfrak{n}_\mathcal{C})$ . By direct calculation, this happens if and only if  $X_{\gamma,k}(f_{\alpha,i}) = 0$ , or equivalently if and only if

$$\frac{\partial}{\partial x_{\gamma,k}}(f_{\alpha,i}) = 0$$

for every  $\alpha \in \mathcal{R}$  and  $\gamma \in \mathcal{C}$ .

The map  $\nu$  is a homomorphism because  $\tau$  and the projection operator are such. Hence  $\nu(\mathfrak{q})$  is a Lie algebra of multicontact vector fields on  $S$ .

We now investigate the kernel of  $\nu$  in the case  $\Delta \subset \mathcal{R}$ . Since  $\tau(E) = \sum_{\gamma \in \mathcal{C}} \sum_{k=1}^{m_\gamma} g_{\gamma,k} X_{\gamma,k}$  for every  $E \in \mathfrak{n}_{\mathcal{C}}$ , the inclusion  $\mathfrak{n}_{\mathcal{C}} \subseteq \ker \nu$  follows. We prove the opposite inclusion by treating separately each component of  $E \in \ker \nu$ , written according to the following vector space direct sum:

$$\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n} \oplus (\bar{\mathfrak{n}} \cap \mathfrak{q}).$$

From now until the end of this proof we write  $n = \exp(W) = \exp(\sum_{\alpha \in \Sigma_+} W_\alpha)$ , where  $W_\alpha \in \mathfrak{g}_\alpha$ .

If  $E \in \mathfrak{n} \cap \ker \nu$ , then  $\overline{\tau(E)} = 0$ . Write  $E = \sum_{\gamma \in \Sigma_+} \sum_{k=1}^{m_\gamma} a_{\gamma,k} E_{\gamma,k}$  and compute

$$\begin{aligned} \tau(E)f &= \frac{d}{dt} f(\exp(-tE)n) \Big|_{t=0} \\ &= \frac{d}{dt} f(\exp(-tE + W - \frac{t}{2}[E, W] + \dots)) \Big|_{t=0}. \end{aligned}$$

If  $E$  were not in  $\mathfrak{n}_{\mathcal{C}}$ , there would exist  $\beta \in \mathcal{R}$  and  $j = 1, \dots, m_\beta$  such that  $a_{\beta,j} \neq 0$ . If  $f : n \mapsto x_{\beta,j}$  then we have that  $\tau(E)f$  is a polynomial in  $\{x_{\alpha,i}\}_{\alpha \in \Sigma_+}$  whose term of degree zero is  $a_{\beta,j}$ . On the other hand

$$\tau(E)x_{\beta,j} = 0 \quad \forall \beta \in \mathcal{R},$$

because its decomposition on the basis of left invariant vector fields involves only components corresponding to the roots in  $\mathcal{C}$ . This is a contradiction.

Let  $E \in \mathfrak{a} \cap \ker \nu$ . Recalling that we view  $N$  as a dense subset of  $G/P$  and that  $\exp(tE) \in P$ , we have

$$\begin{aligned} \tau(E)f(n) &= \frac{d}{dt} f(\exp(-tE)n) \Big|_{t=0} \\ &= \frac{d}{dt} f(\exp(-tE)n \exp(tE)) \Big|_{t=0} \\ &= \frac{d}{dt} f(\exp(\sum_{\alpha \in \Sigma_+} e^{-t\alpha(E)} W_\alpha)) \Big|_{t=0}. \end{aligned}$$

Choose now  $f : n \mapsto x_{\gamma,j}$ , so that

$$\tau(E)f(n) = \frac{d}{dt} (e^{-t\gamma(E)} x_{\gamma,j}) \Big|_{t=0} f(n) = -\gamma(E) x_{\gamma,j}.$$

This is zero for every  $\gamma \in \mathcal{R}$  because  $E$  is in the kernel of  $\nu$ , so that  $\gamma(E) = 0$  for every  $\gamma \in \mathcal{R}$ . Since  $\mathcal{R} \supset \Delta$  and  $\Delta$  is a basis of  $\mathfrak{a}^*$ , the dual space of  $\mathfrak{a}$ , it follows that  $E = 0$ .

Let  $E \in \mathfrak{m} \cap \ker\nu$ . Since  $\mathfrak{m}$  normalizes every root space, if  $f : n \mapsto x_{\gamma,j}$ , then

$$\begin{aligned}\tau(E)f(n) &= \frac{d}{dt}f(\exp(e^{-\text{ad}tE}W))\Big|_{t=0} \\ &= \frac{d}{dt}f(\exp(\sum_{\alpha \in \Sigma_+} \sum_{n=1}^{\infty} (-1)^n t^n \frac{(\text{ad}E)^n}{n!} W_\alpha))\Big|_{t=0} \\ &= ((-\text{ad}E)W_\gamma)_j.\end{aligned}$$

Whenever  $\gamma \in \mathcal{R}$  we have  $((-\text{ad}E)W_\gamma)_j = 0$  for every  $j$ . Thus  $(\text{ad}E)\mathfrak{g}_\gamma = 0$  for every  $\gamma \in \mathcal{R}$ . In particular  $(\text{ad}E)\mathfrak{g}_\delta = 0$  for every simple root  $\delta$ , and Jacobi identity implies  $(\text{ad}E)\mathfrak{n} = 0$ . Since  $\theta E = E$ , it follows that  $(\text{ad}E)\mathfrak{g}_{-\delta} = (\text{ad}\theta E)\mathfrak{g}_\delta = (\text{ad}E)\mathfrak{g}_\delta = 0$ . Hence  $(\text{ad}E)\mathfrak{g} = 0$ . Thus  $E \in Z(\mathfrak{g}) = \{0\}$ .

Let now  $E \in \mathfrak{g}_\beta \cap \mathfrak{q} \cap \ker\nu$  for some negative root  $\beta$ , so that  $\overline{\tau(E)} = 0$ . For every  $E' \in \mathfrak{n}$  we have

$$[\tau(E), \tau(E')] = [\sum_{\alpha \in \mathcal{C}} \sum_{i=1}^{m_\alpha} f_{\alpha,i} X_{\alpha,i}, \sum_{\beta \in \mathcal{R}} \sum_{j=1}^{m_\beta} g_{\beta,j} X_{\beta,j} + \sum_{\gamma \in \mathcal{C}} \sum_{k=1}^{m_\gamma} g_{\gamma,k} X_{\gamma,k}]$$

All terms of the bracket above lie on  $\mathfrak{n}_{\mathcal{C}}$ , except for summands of the form

$$f_{\alpha,i} X_{\alpha,i} (g_{\beta,j}) X_{\beta,j},$$

but  $X_{\alpha,i}(g_{\beta,j}) = 0$ , for every  $\alpha \in \mathcal{C}$  and  $\beta \in \mathcal{R}$ , because the coefficients  $g_{\beta,j}$  are  $\mathcal{C}$ -independent. It follows in particular that

$$\overline{[\tau(E), \tau(E')]} = 0,$$

thus  $[E, E'] \in \ker\nu$  for every  $E' \in \mathfrak{n}$ . Therefore one can chose  $E'$  such that  $[E, E'] \in \mathfrak{m} \oplus \mathfrak{a}$ . But this is a contradiction, because no elements of  $\mathfrak{m} \oplus \mathfrak{a}$  lie in the kernel of  $\nu$ .  $\square$

Notice that in the last step of the above proof we did not use that the negative root is in the normalizer, so that there are no multicontact vector fields on  $N$  coming from a negative root that become zero once projected to a slice representing a Hessenberg manifold.

### 3. Iwasawa sub-models

The converse of Theorem 20 is true under the hypothesis (I) of the Theorem 22 below. We remind the reader that by Theorem 15 we are assuming that  $\mathcal{R}$  consists of a single dark zone and that it contains all the simple restricted roots.

LEMMA 21. *If the vector space*

$$\mathfrak{n}^\mu = \bigoplus_{\alpha \in \mathcal{S}_\mu} \mathfrak{g}_\alpha$$

*is a subalgebra of  $\mathfrak{n}$ , then in particular it is an Iwasawa nilpotent Lie algebra.*

PROOF. The algebra  $\mathfrak{n}^\mu$  coincides with the nilpotent algebra generated by the root spaces corresponding to the simple roots in  $\mathcal{S}_\mu$ . Hence it is an Iwasawa Lie algebra because it is the canonical nilpotent algebra associated to a connected Dynkin diagram, together with admissible multiplicity data.  $\square$

The following theorem holds.

THEOREM 22. *Let  $\mathfrak{g}$  be a simple Lie algebra of real rank strictly greater than two and  $\mathcal{R} \subset \Sigma_+$  a subset of Hessenberg type satisfying*

- (I) *each shadow in the Hessenberg set defines a subalgebra of  $\mathfrak{n}$ ,*
- (II) *each shadow contains at least two simple roots.*

*Then the Lie algebra of multicontact vector fields on  $\text{Hess}_{\mathcal{R}}(H)$  is isomorphic to  $\mathfrak{q}/\mathfrak{n}_{\mathcal{C}}$ , for every regular element  $H \in \mathfrak{a}$  and where  $\mathfrak{q} = N_{\mathfrak{g}}\mathfrak{n}_{\mathcal{C}}$ .*

Hypothesis (II) just avoids the rank one cases. More precisely, if (II) is not true, then  $\mathcal{R}$  defines a rank one Iwasawa subalgebra. In this case, however, the finite dimensionality of the Lie algebra  $MC(S)$  is no longer guaranteed<sup>1</sup>.

We have seen that if  $\mathcal{R}$  is an arbitrary Hessenberg type set, then all elements in  $\mathfrak{q}/\mathfrak{n}_{\mathcal{C}}$  define multicontact vector fields. In order to show the converse, we look again at the system of differential equation (41). If  $F \in MC(S)$ , then its coefficients solve (41), and they solve all the subsystems that we can extract from (41). Therefore we obtain necessary conditions by looking at some special subsystems of (41). In particular we consider a

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<sup>1</sup>Personal communication by the authors of [12], who intend to clarify this matter in full detail in a forthcoming paper.

subsystem for each shadow, namely:

$$(48) \quad \begin{cases} X_{\delta,i}(f_{\gamma,j}) = 0 & \delta \in S_\mu, \text{ if } \gamma - \delta \notin \Sigma_+ \cup \{0\} \\ X_{\delta,i}(f_{\gamma,j}) + \sum_{l=1}^{m_{\gamma-\delta}} c_{\delta,\gamma-\delta}^{ilj} f_{\gamma-\delta,l} = 0 & \text{if } \gamma - \delta \in \Sigma_+ \\ X_{\delta,i}(f_{\gamma,j}) = 0 & \delta \notin S_\mu, \end{cases}$$

for every root  $\gamma$  in  $S_\mu$ . Here we stress that the functions  $f_{\gamma,j}$  are defined on some open subset of the slice  $S$ . Since each shadow defines an Iwasawa Lie algebra, the system above looks like the system of differential equations that defines the multicontact vector fields on nilpotent Iwasawa Lie groups. We want to interpret (48) exactly in this way. Indeed, Lemma 19 tells us that the functions  $f_{\gamma,j}$ , as  $\gamma$  varies in  $S_\mu$ , are  $(\Sigma_+ \setminus S_\mu)$ -independent. Hence

$$X_{\delta,i}(f_{\gamma,j}) = X_{\delta,i}^\mu(f_{\gamma,j}), \text{ for every } \gamma, \delta \in S_\mu,$$

where  $X_{\delta,i}^\mu$  is the vector field that is obtained from  $X_{\delta,i}$  by setting to zero all the components that are labeled by roots that are not in  $S_\mu$ . We then consider, in place of (48), the equivalent system

$$(49) \quad \begin{cases} X_{\delta,i}^\mu(f_{\gamma,j}) = 0 & \text{if } \gamma - \delta \notin \Sigma_+ \cup \{0\} \\ X_{\delta,i}^\mu(f_{\gamma,j}) + \sum_{l=1}^{m_{\gamma-\delta}} c_{\delta,\gamma-\delta}^{ilj} f_{\gamma-\delta,l} = 0 & \text{if } \gamma - \delta \in \Sigma_+, \end{cases}$$

where  $\gamma, \delta \in S_\mu$ . Define  $\mathfrak{n}^\mu$  as in Lemma 21. Using hypothesis (I) of Theorem 22, Lemma 21 implies that the Lie algebra  $\mathfrak{n}^\mu$  is an Iwasawa nilpotent Lie algebra. The system of differential equations above coincides with the multicontact conditions for a vector field on  $N^\mu = \exp \mathfrak{n}^\mu$ , because the vector fields  $X_{\delta,i}^\mu$  are exactly the left-invariant vector fields on  $N^\mu$ . This latter assertion is rather obvious, and it is a consequence of a direct calculation similar to the one involved in Lemma 12. Summarizing, we have the following result.

**PROPOSITION 23.** *Let  $F \in \mathfrak{X}(S)$ . Then  $F \in MC(S)$  if and only if its projection*

$$F^\mu = \sum_{\alpha \in S_\mu} \sum_{i=1}^{m_\alpha} f_{\alpha,i} \overline{X}_{\alpha,i},$$

is a multicontact vector field on  $N^\mu$  for every maximal root  $\mu$ .

PROOF. “ $\Rightarrow$ ”. By Lemma 19, any multicontact vector field on  $S$  can be naturally viewed as a vector field on  $N^\mu$  for every maximal root  $\mu$ . If the coefficients of  $F$  solve the system of differential equations (41), then in particular they solve all subsystems (49), that is any projected vector field  $F^\mu$  is in  $MC(N^\mu)$ .

“ $\Leftarrow$ ”. If  $F$  has the property that each  $F^\mu$  solves (49), then  $F$  solves all the equations in (41), so that it is in  $MC(S)$ .  $\square$

The multicontact vector fields on a nilpotent Iwasawa Lie group are studied in [12], where it is showed in particular that the Lie algebra of such vector fields on  $N^\mu$  is isomorphic to  $\mathfrak{g}^\mu = \mathfrak{n}^\mu + \theta\mathfrak{n}^\mu + \mathfrak{m}^\mu + \mathfrak{a}^\mu$ , where

$$\mathfrak{m}^\mu = \mathfrak{m} \cap [\mathfrak{n}^\mu, \theta\mathfrak{n}^\mu],$$

and

$$\mathfrak{a}^\mu = \mathfrak{a} \cap [\mathfrak{n}^\mu, \theta\mathfrak{n}^\mu].$$

In fact each element of this algebra defines a vector field on  $N^\mu$  whose coefficients solve (48). These vector fields are realized as  $\tau_\mu(E)$ , where

$$\tau_\mu(E)f(n) = \frac{d}{dt}f(\exp(-tE)n)\Big|_{t=0},$$

with  $E \in \mathfrak{g}^\mu$ ,  $n \in N^\mu$  and some function  $f$  on  $N^\mu$ . Since  $\mathfrak{g}^\mu$  is a subalgebra of  $\mathfrak{g}$ , it is possible to see the multicontact vector fields on  $N^\mu$  as the projections of suitable vector fields on  $N$ , in the sense that is explained in the following proposition.

PROPOSITION 24. *The set of vector fields*

$$\{\tau(E)^\mu, E \in \mathfrak{g}^\mu\}$$

*generates the Lie algebra  $MC(N^\mu)$ , where*

$$\tau(E)^\mu = \sum_{\gamma \in \mathcal{S}_\mu} \sum_{j=1}^{m_\gamma} f_{\gamma,j} \overline{X}_{\gamma,j},$$

*whenever  $\tau(E) = \sum_{\gamma \in \Sigma_+} \sum_{j=1}^{m_\gamma} f_{\gamma,j} X_{\gamma,j}$ . In particular, if  $E \in \mathfrak{q}$ , it follows that  $\tau(E)^\mu \neq 0$  if and only if  $E \in \mathfrak{g}^\mu \setminus \{0\}$ .*

PROOF. Let  $E \in \mathfrak{g}^\mu$ . We show that  $E \in \mathfrak{b}$ , the normalizer in  $\mathfrak{g}$  of the nilpotent ideal consisting of all the root spaces labeled by  $\mathcal{S}_\mu^c = \Sigma_+ \setminus \mathcal{S}_\mu$ , namely  $\mathfrak{b} = N_{\mathfrak{g}}\mathfrak{n}_{\mathcal{S}_\mu^c}$ . Since  $\mathfrak{g}^\mu = \mathfrak{m}^\mu + \mathfrak{a}^\mu + \mathfrak{n}^\mu + \theta\mathfrak{n}^\mu$  and  $\mathfrak{m}^\mu + \mathfrak{a}^\mu + \mathfrak{n}^\mu \subseteq \mathfrak{b}$ , we can suppose that  $E \in \theta\mathfrak{n}^\mu$ . Write  $E = \sum E_\beta$ . If  $E \notin \mathfrak{b}$ , then there

exists  $\beta$  such that  $E_\beta \notin \mathfrak{b}$ . In this case, since  $\mathfrak{b}$  normalizes, there would exist  $\alpha \in \mathcal{S}_\mu^c$  such that  $\alpha + \beta \notin \mathcal{S}_\mu$ —a contradiction, because the sum of two roots in  $\mathcal{S}_\mu$  is in  $\mathcal{S}_\mu$ , as follows from hypothesis (I). Theorem 20 applied to the Hessenberg set  $\mathcal{S}_\mu$  implies that  $\tau(E)^\mu \in MC(\mathbf{N}^\mu)$ .

We now show that the vector fields  $\tau(E)^\mu$  are all different from zero whenever  $E \in \mathfrak{g}^\mu \setminus \{0\}$ . Suppose that there exists  $E \in \mathfrak{g}^\mu$  such that  $\tau(E)^\mu = 0$ . Write  $E = H + K + \sum E_\alpha$ , with  $H \in \mathfrak{a}^\mu$  and  $K \in \mathfrak{m}^\mu$ . Since  $Y \mapsto Y^\mu$  preserves (homomorphic images of) root spaces, the hypothesis  $\tau(E)^\mu = 0$  is equivalent to assuming  $\tau(H)^\mu = \tau(K)^\mu = 0$  and  $\tau(E_\alpha)^\mu = 0$  for every  $\alpha$ . We show first that  $H = 0$ . Indeed, writing  $n = \exp(\sum_{\alpha \in \Sigma_+} W_\alpha)$ , we have

$$\begin{aligned}\tau(H)f(n) &= \frac{d}{dt} f(\exp(-tH)n) \Big|_{t=0} \\ &= \frac{d}{dt} f(\exp(\sum_{\alpha \in \Sigma_+} e^{-t\alpha(H)} W_\alpha)) \Big|_{t=0}.\end{aligned}$$

Hence, if  $f : n \mapsto x_{\gamma,j}$ , then

$$\tau(H)f(n) = \frac{d}{dt} (e^{-t\gamma(H)} x_{\gamma,j}) f(n) \Big|_{t=0} = -\gamma(H) x_{\gamma,j}.$$

This is zero for every  $\gamma$  in  $S_\mu$ , and in particular  $\delta(H) = 0$  for every simple roots  $\delta$  in  $S_\mu$ . By duality this implies that  $H = 0$ .

Suppose now that  $\tau(K)^\mu = 0$ . Since  $\mathfrak{m}^\mu$  normalizes every root space, if  $f : n \mapsto x_{\gamma,j}$ , then

$$\begin{aligned}\tau(K)f(n) &= \frac{d}{dt} f(\exp(e^{-\text{ad}tK} W)) \Big|_{t=0} \\ &= \frac{d}{dt} f(\exp(\sum_{\alpha \in \Sigma_+} \sum_{n=1}^{\infty} (-1)^n t^n \frac{(\text{ad}K)^n}{n!} W_\alpha)) \Big|_{t=0} \\ &= ((-\text{ad}K)W_\gamma)_j.\end{aligned}$$

Whenever  $\gamma \in \mathcal{S}_\mu$  we have  $((-\text{ad}K)W_\gamma)_j = 0$  for every  $j$ . Thus  $(\text{ad}K)\mathfrak{g}_\gamma = 0$  for every  $\gamma \in \mathcal{S}_\mu$ . In particular  $(\text{ad}K)\mathfrak{g}_\delta = 0$  for every simple root  $\delta \in \mathcal{S}_\mu$ , and Jacobi identity implies  $(\text{ad}K)\mathfrak{n}^\mu = 0$ . Since  $\theta K = K$ , it follows that  $(\text{ad}K)\mathfrak{g}_{-\delta} = (\text{ad}\theta K)\mathfrak{g}_\delta = (\text{ad}K)\mathfrak{g}_\delta = 0$ . Hence  $(\text{ad}K)\mathfrak{g}^\mu = 0$ . Thus  $K \in Z(\mathfrak{g}^\mu) = \{0\}$ .

Next, suppose that  $\tau(E_\alpha)^\mu = 0$ . Then

$$0 = [\tau(E_\alpha)^\mu, \tau(\theta E_\alpha)^\mu] = \tau([E_\alpha, \theta E_\alpha])^\mu.$$

Recall that if  $E_\alpha \neq 0$ , then

$$(50) \quad [E_\alpha, \theta E_\alpha] = B(E_\alpha, \theta E_\alpha) H_\alpha$$

(see e.g. Prop 6.52 in [21]), where  $H_\alpha \in \mathfrak{a}$  represents  $\alpha$  via the Killing form and  $B(E_\alpha, \theta E_\alpha) < 0$ . But  $0 = \tau([E_\alpha, \theta E_\alpha])^\mu = \tau(H_\alpha)$  implies  $H_\alpha = 0$  by the previous case, contradicting (50).

Therefore the set

$$\{\tau(E)^\mu, E \in \mathfrak{g}^\mu\}$$

generates the Lie algebra of multicontact vector fields on  $N^\mu$ , as required.

Finally, let  $E \in \mathfrak{q}$ . We proved above that if  $E \in \mathfrak{g}^\mu$  then  $\tau(E)^\mu \neq 0$ . On the other hand,  $E \in \mathfrak{q}$  implies that  $\overline{\tau(E)} \in MC(S)$ , so that in particular  $\tau(E)^\mu \in MC(N^\mu)$ . If  $E \notin \mathfrak{g}^\mu$ , then the latter assertion is possible only if  $\tau(E)^\mu = 0$ .  $\square$

We prove a result that comes as a direct consequence of Proposition 24. It essentially says that the intersection of two or more shadows still defines an Iwasawa nilpotent subalgebra, and that we can represent the multicontact vector fields on the corresponding subgroup again by projecting some multicontact vector field of the form  $\tau(E)$ . This result will be used later, and it can be viewed as a generalization of the extra-cross condition (36) of the Example 1.2.

**COROLLARY 25.** *Let  $\mathcal{I} = \bigcap_{\mu \in \mathcal{E}} \mathcal{S}_\mu$  with  $\mathcal{E}$  a subset of maximal roots in  $\mathcal{R}$ . Then:*

- (i) *the nilpotent Lie algebra  $\mathfrak{n}^{\mathcal{I}} = \bigoplus_{\alpha \in \mathcal{I}} \mathfrak{g}_\alpha$  is an Iwasawa Lie algebra.*
- (ii) *Let  $\mathfrak{g}^{\mathcal{I}}$  denote the Lie subalgebra of  $\mathfrak{g}$  generated by  $\mathfrak{n}^{\mathcal{I}}$  and  $\theta \mathfrak{n}^{\mathcal{I}}$ , and let  $N^{\mathcal{I}} = \exp \mathfrak{n}^{\mathcal{I}}$ . The vector fields of the type*

$$\tau(E)^{\mathcal{I}} = \sum_{\alpha \in \mathcal{I}} \sum_{j=1}^{m_\gamma} f_{\gamma,j} \overline{X}_{\gamma,j},$$

*with  $E \in \mathfrak{g}^{\mathcal{I}}$ , are in  $MC(N^{\mathcal{I}})$ .*

- (iii) *If  $E \in \mathfrak{q}$ , then  $E \in \mathfrak{g}^{\mathcal{I}} \setminus \{0\}$  implies that  $\tau(E)^{\mathcal{I}} \neq 0$ .*

**PROOF.** (i) Let  $\alpha$  and  $\beta$  two roots in  $\mathcal{I}$  such that  $\alpha + \beta$  is a root. Then  $\alpha + \beta \in \mathcal{S}_\mu$  for every  $\mu \in \mathcal{E}$ , because each shadow defines a subalgebra. This implies that  $\alpha + \beta \in \mathcal{I}$ , so that  $\mathfrak{n}^{\mathcal{I}}$  is a subalgebra in  $\mathfrak{n}$ . By Lemma 21,  $\mathfrak{n}^{\mathcal{I}}$  is an Iwasawa nilpotent Lie algebra.

(ii) Because of (i), we can use the results in [12] in order to describe the multicontact vector fields on  $N^{\mathcal{I}}$ . Thus, the same argument as in the proof of Proposition 24 with  $\mathcal{I}$  in place of  $\mathcal{S}_\mu$  shows that  $\tau(E)^{\mathcal{I}} \in MC(N^{\mathcal{I}})$ .

(iii) See the proof of Proposition 24.  $\square$

Notice that (ii) of the previous corollary asserts  $\tau(\mathfrak{g}^{\mathcal{I}}) \subset MC(N^{\mathcal{I}})$  and does not claim equality. This is because the intersection  $\mathcal{I}$  may well give rise to rank-one algebras, so that the results of [12] do not apply to prove the reverse inclusion. Similarly, (iii) gives only one implication.

We shall conclude the proof of Theorem 22 by setting up a diagram of the following type

$$(51) \quad \begin{array}{ccc} MC(S) & \xrightarrow{k} & \mathfrak{g}^1 \oplus \cdots \oplus \mathfrak{g}^p \\ & \searrow & \uparrow \bar{\ell} \\ & & \mathfrak{q}/\mathfrak{n}_C \end{array}$$

and then showing that both  $k$  and  $\bar{\ell}$  are injective linear maps with equal images. Thus  $MC(S)$  and  $\mathfrak{q}/\mathfrak{n}_C$  will be seen to be isomorphic vector spaces. Moreover, the induced left arrow coincides with the quotient map of the map  $E \mapsto \overline{\tau(E)}$  defined on  $\mathfrak{q}$ , which is a Lie algebra homomorphism. This latter fact is a straightforward consequence of the definition of the various maps, that we illustrate below.

Fix a numbering  $\mu_1, \dots, \mu_p$  of the maximal roots and write  $\mathfrak{g}^i$  for  $\mathfrak{g}^{\mu_i}$ . By Proposition 23, we can associate to each  $F \in MC(S)$  a vector  $(F^1, \dots, F^p)$ , where each  $F^i = F^{\mu_i} \in MC(N^{\mu_i})$  is the natural projection. Moreover, by Proposition 24, we know that  $F^i = \tau(E)^i$  for some  $E \in \mathfrak{g}^i$ . These observations allow us to define a map

$$k : MC(S) \longrightarrow \mathfrak{g}^1 \oplus \cdots \oplus \mathfrak{g}^p$$

by setting  $k(F) = (F^1, \dots, F^p)$ . Here a few comments are in order. First of all, the direct sum on the right is viewed merely as a vector space. Secondly, as already observed, we identify each  $\mathfrak{g}^{\mu}$  with  $\{\tau(E)^{\mu}, E \in \mathfrak{g}^{\mu}\}$ , by means of Proposition 24. Finally, the map  $k$  is injective because  $(F^1, \dots, F^p)$  encodes all components of  $F$ .

Next, observe that the assignment

$$E \mapsto (\tau(E)^1, \dots, \tau(E)^p)$$

gives a well-defined map

$$\ell : \mathfrak{q} \longmapsto \mathfrak{g}^1 \oplus \cdots \oplus \mathfrak{g}^p$$

because by Proposition 24  $\tau(E)^i \neq 0$  only if  $E \in \mathfrak{g}^i$ . The kernel of  $\ell$  is  $\mathfrak{n}_C$  because if all  $\tau(E)^i$  vanish, then so does  $\overline{\tau(E)}$ , and Theorem 20 says  $E \in \mathfrak{n}_C$ . Therefore  $\ell$  projects to a map  $\bar{\ell}$  defined on  $\mathfrak{q}/\mathfrak{n}_C$ .

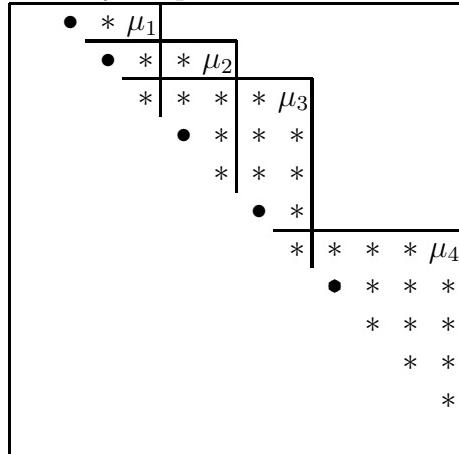
All the ingredients appearing in the diagram (51) have been defined and all we need to show is that  $k(MC(S)) = \ell(\mathfrak{q}/\mathfrak{n}_c)$ . This will be formalized in Proposition 28 below. Its proof needs a technical lemma that characterizes  $\mathfrak{q}$  in terms of roots. We underscore that this characterization (i.e. Lemma 27) holds *only* under the hypothesis (I) in Theorem 22. Before we set up all the needed machinery, we point out the main issue. Recall that  $\mathfrak{q}$  is the normalizer of the nilpotent ideal  $\mathfrak{n}_c$  in  $\mathfrak{g}$ , so that, as we already observed, it clearly contains  $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ . Breaking it according to the root space decomposition of  $\mathfrak{g}$ , for short  $\mathfrak{g} = \overline{\mathfrak{n}} \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ , we may write  $\mathfrak{q} = (\mathfrak{q} \cap \overline{\mathfrak{n}}) \oplus (\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n})$ . Since  $\mathfrak{q}$  is a parabolic subalgebra of  $\mathfrak{g}$ , then  $\mathfrak{q} \cap \overline{\mathfrak{n}} = \sum_{\alpha \in \mathcal{D}} \mathfrak{g}_\alpha$ , for some  $\mathcal{D} \subset \Sigma_-$  (see [21], Sec.7, Ch.VII). The point addressed by Lemma 27 is an adequate description of  $\mathfrak{q} \cap \overline{\mathfrak{n}}$ .

Given a root  $\alpha = \sum_{\delta \in \Delta} n_\delta(\alpha) \delta$  we denote by  $\mathcal{Y}(\alpha)$  the subset of  $\Delta$  consisting of those  $\delta$  for which  $n_\delta(\alpha) \neq 0$ , and we call it the *simple support* of  $\alpha$ . For later use, we recall the following result (see Corollary 3, Ch. VI, 1, pag. 160 in [Bourbaki]).

**PROPOSITION 26** (Bourbaki).

- (i) *Let  $\alpha \in \Sigma$ . Then  $\mathcal{Y}(\alpha)$  is a connected subset of the Dynkin diagram associated to  $\Sigma$ .*
- (ii) *Let  $\mathcal{Y}$  be any connected non empty subset of a Dynkin diagram. Then  $\sum_{\beta \in \mathcal{Y}} \beta$  is a root.*

Yet another piece of notation. We say that a simple root  $\delta$  is a *boundary simple root* if there exists a maximal root  $\nu$  in  $\mathcal{R}$  whose simple support is a connected diagram that does not contain  $\delta$  but to which  $\delta$  is adjacent, i.e. such that there exists  $\delta' \in \mathcal{Y}(\nu)$  with the property that  $\delta + \delta'$  is a root. The set of all the boundary simple roots will be denoted by  $\mathcal{B}$ .



In the picture above the dark dots are the boundary roots. Each shadow determines a connected line in the  $A_n$ -type Dynkin diagram, adjacent to which lie exactly two boundary roots if the line is inside the diagram, otherwise just one. (Recall that this refers to  $\mathfrak{sl}(n, \mathbb{R})$ ).

LEMMA 27. *Let  $\mathfrak{q} \cap \bar{\mathfrak{n}} = \sum_{\alpha \in \mathcal{D}} \mathfrak{g}_\alpha$ .*

- (i) *If  $\delta$  is a simple root, then  $-\delta \notin \mathcal{D}$  if and only if  $\delta \in \mathcal{B}$ .*
- (ii) *If  $\alpha$  is any positive root, then  $-\alpha \notin \mathcal{D}$  if and only if the simple support of  $\alpha$  contains a simple root in  $\mathcal{B}$ .*

PROOF. We prove (i) first.

“ $\Leftarrow$ ”. Let  $\delta \in \mathcal{B}$  and let  $\nu$  be a maximal root to whose shadow  $\delta$  is adjacent. Proposition 26 implies that  $\sum_{\varepsilon \in \mathcal{Y}(\nu)} \varepsilon + \delta = \sigma + \delta$  is a root. Moreover, it does not lie in  $\mathcal{R}$ . Indeed, if  $\sigma + \delta \in \mathcal{R}$ , then it would belong to a shadow containing  $\mathcal{S}_\nu$ , contradicting the maximality of  $\nu$ . On the other hand,  $\sigma$  itself is a root, again by Proposition 26, and it lies in  $\mathcal{R}$ , because it is sum of simple roots in a same shadow  $\mathcal{S}_\nu$ . Thus, we found a root in  $\mathcal{C}$ , namely  $\sigma + \delta$ , such that  $(\sigma + \delta) - \delta \notin \mathcal{C}$ . Therefore  $-\delta \notin \mathcal{D}$ .

“ $\Rightarrow$ ”. Suppose  $\delta \notin \mathcal{B}$ . Let  $\alpha \in \mathcal{C}$  with  $\delta \prec \alpha$  and consider its simple support  $\mathcal{Y}(\alpha)$ . We shall show that  $\alpha - \delta \in \mathcal{C}$  whenever  $\alpha - \delta \in \Sigma$ . Take a maximal connected set  $\mathcal{F}$  of simple roots in  $\mathcal{Y}(\alpha)$  with the following properties:

- $\diamond \delta \in \mathcal{F};$
- $\diamond$  there exists a shadow containing  $\mathcal{F}$ .

This means that  $\delta \in \mathcal{F} \subset \mathcal{S}_\nu$  for some  $\nu$ , but no larger connected subset of  $\mathcal{Y}(\alpha)$  containing  $\delta$  is contained in any other single shadow. Necessarily  $\mathcal{F}$  is a proper subset of  $\mathcal{Y}(\alpha)$ , for otherwise  $\alpha$  would lie in  $\mathcal{R}$ . Take  $\varepsilon \in \mathcal{Y}(\alpha)$  adjacent to  $\mathcal{F}$ . Then two cases arise.

- (a)  $\mathcal{Y}(\alpha - \delta)$  does contain  $\delta$ . In this case  $\mathcal{Y}(\alpha - \delta)$  contains both  $\mathcal{F}$  and  $\varepsilon$ . Thus  $\alpha - \delta \notin \mathcal{R}$ , for otherwise  $\mathcal{F} \cup \{\varepsilon\}$  would be a connected set contained in a single shadow (namely any shadow containing  $\alpha - \delta$ ) and it would be larger than  $\mathcal{F}$ .
- (b)  $\mathcal{Y}(\alpha - \delta)$  does not contain  $\delta$ . Then  $\mathcal{Y}(\alpha - \delta)$  is connected and  $\delta$  is adjacent to it. If  $\alpha - \delta \in \mathcal{R}$  then  $\delta$  would be a boundary root because  $\mathcal{Y}(\alpha - \delta) \subset \mathcal{S}_\nu$  for some maximal root  $\nu$ , and  $\delta \notin \mathcal{S}_\nu$  (for otherwise  $\alpha = (\alpha - \delta) + \delta \in \mathcal{S}_\nu$ , which is impossible). Hence  $\delta$  would be adjacent to the simple support of  $\mathcal{S}_\nu$ , contradicting  $\delta \notin \mathcal{B}$ . Therefore  $\alpha - \delta \notin \mathcal{R}$ .

We have seen that in all cases  $-\delta \in \mathcal{D}$ . This concludes the proof of (i).

As for (ii), take a non simple root  $-\alpha \notin \mathcal{D}$ . Then  $\mathcal{Y}(\alpha)$  contains at least one simple root  $\delta \notin -\mathcal{D}$ . Indeed, since  $\mathfrak{q}$  is a subalgebra, if  $\mathcal{Y}(\alpha)$  were contained in  $-\mathcal{D}$ , then  $\alpha$  itself would lie in  $\mathfrak{q}$ . Thus  $\mathcal{Y}(\alpha)$  contains a boundary simple root. Conversely, if  $\alpha \in \Sigma_+$  is such that  $\mathcal{Y}(\alpha)$  contains a simple root in  $\mathcal{B}$ , then it contains a simple that is not  $-\mathcal{D}$ , so that  $-\alpha$  is not in  $\mathcal{D}$ .  $\square$

**PROPOSITION 28.** *In the notations of diagram (51)*

$$(F^1, \dots, F^p) \in k(MC(S)) \Leftrightarrow (F^1, \dots, F^p) = \ell(E) \text{ for some } E \in \mathfrak{q}.$$

**PROOF.** “ $\Leftarrow$ ”. Let  $E \in \mathfrak{q}$ . Then  $\ell(E) = (\tau(E)^1, \dots, \tau(E)^p)$ . By Theorem 20, it follows that  $\overline{\tau(E)} \in MC(S)$ . Therefore

$$k(\overline{\tau(E)}) = (\tau(E)^1, \dots, \tau(E)^p) = \ell(E).$$

“ $\Rightarrow$ ”. Let  $F \in MC(S)$  and  $k(F) = (F^1, \dots, F^p)$ . Proposition 24 implies that for all  $i = 1, \dots, p$  there exists  $E^i \in \mathfrak{g}^i$  such that  $F^i = \tau(E^i)^i$ . Write  $E^i = \sum_{\alpha \in \Sigma^i \cup \{0\}} E_\alpha^i$ , with  $\Sigma^i = \mathcal{S}_{\mu_i} \cup (-\mathcal{S}_{\mu_i})$ . By definition,  $\tau(E^i)^i \in MC(N^{\mu_i})$  if and only if  $\tau(E_\alpha^i)^i \in MC(N^{\mu_i})$  for every  $\alpha \in \Sigma^i \cup \{0\}$ .

Recall that  $\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n} \oplus (\overline{\mathfrak{n}} \cap \mathfrak{q})$ , and write  $\mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{a}$ ,  $\mathfrak{n} = \bigoplus_{\gamma \in \Sigma_+} \mathfrak{g}_\gamma$  and  $\overline{\mathfrak{n}} \cap \mathfrak{q} = \bigoplus_{\gamma \in \mathcal{D}} \mathfrak{g}_\gamma$ . Therefore, the normalizer can be written as follows:

$$\mathfrak{q} = \bigoplus_{\alpha \in \mathcal{G}} \mathfrak{g}_\alpha,$$

where  $\mathcal{G} = \Sigma_+ \cup \{0\} \cup \mathcal{D}$ . Using these notations, we shall prove the following two claims:

- (a)  $\alpha \in \mathcal{G} \Rightarrow E_\alpha^i = E_\alpha^j$ , for every  $i, j$ ;
- (b)  $\alpha \notin \mathcal{G} \Rightarrow E_\alpha^i = 0$ .

These two facts allows us to define an element  $E = \sum_{\alpha \in \Sigma \cup \{0\}} E_\alpha$  by

$$E_\alpha = \begin{cases} E_\alpha^i & \text{if } \alpha \in \mathcal{G} \\ 0 & \text{if } \alpha \notin \mathcal{G}, \end{cases}$$

for all  $i = 1, \dots, p$ . In particular,  $E \in \mathfrak{q}$  and  $\ell(E) = (F^1, \dots, F^p)$ , that proves the proposition.

(a) If  $\alpha \in \mathcal{G}$ , then  $E_\alpha^i \in \mathfrak{q}$  for every  $i = 1, \dots, p$ . By Theorem 20,  $\overline{\tau(E_\alpha^i)} \in MC(S)$  and, by Proposition 24,  $\tau(E_\alpha^i) \in MC(N^\mu)$  for every maximal root  $\mu$ . Moreover, Proposition 24 also implies that  $\tau(E_\alpha^i)^j \neq 0$  if and only if  $E_\alpha^i \in \mathfrak{g}^{\mu_j}$ . Suppose that  $E_\alpha^i$  belongs to  $\mathfrak{g}^{\mu_j}$  with  $j \neq i$  and let  $\mathcal{I} = \mathcal{S}_{\mu_i} \cap \mathcal{S}_{\mu_j}$  ( $\mathcal{I}$  is not empty, otherwise  $\mathfrak{g}^{\mu_i}$  and  $\mathfrak{g}^{\mu_j}$  would not have a common element). Then statement (iii) of Corollary 25 implies that the

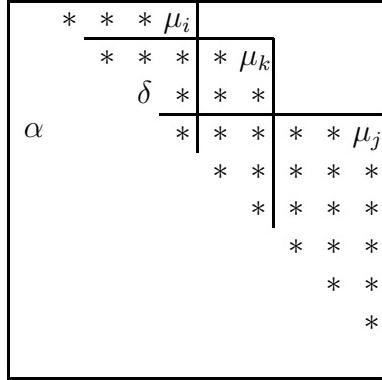
components of  $\tau(E_\alpha^i)^i$  labeled by  $\mathcal{I}$  do not vanish identically. This forces  $F^j \neq 0$ , because

$$\tau(E_\alpha^j)^\mathcal{I} = \tau(E_\alpha^i)^\mathcal{I} \neq 0.$$

Moreover, since  $\mathfrak{g}_\beta \subset \mathfrak{q}$ , the identity  $\tau(E_\alpha^j - E_\alpha^i)^\mathcal{I} = 0$  holds only if  $E_\alpha^i = E_\alpha^j$ , again by (iii) in Corollary 25. This proves (a).

(b). Let  $\alpha \notin \mathcal{G}$ , and suppose that  $E_\alpha^i \neq 0$ . We show that this hypothesis takes us to a contradiction. In particular, we shall show that in the vector  $(F^1, \dots, F^p)$  appears one component that is not of multicontact type, there implying that  $F$  itself is not a multicontact vector field.

By definition of  $\mathcal{G}$ , the root  $\alpha$  must be negative. Furthermore, by (ii) of Lemma 27, there exists  $\delta \in \mathcal{B}$  such that  $\delta + \delta_1 + \dots + \delta_q = -\alpha$ . Let  $\mathcal{S}_{\mu_j}$  be a shadow to which  $\delta$  is adjacent. Then there exists at least a shadow to which  $\delta$  belongs that intersects  $\mathcal{S}_{\mu_j}$ . Indeed, if this does not happen, then  $\delta$  would belong to a dark zone disjoint from  $\mathcal{S}_{\mu_j}$ , which is impossible. Call  $\mathcal{S}_{\mu_k}$  such a shadow and  $\mathcal{J} = \mathcal{S}_{\mu_j} \cap \mathcal{S}_{\mu_k} \neq \emptyset$ .



In the picture above we illustrate, in the case of  $\mathfrak{sl}(n, \mathbb{R})$ , a situation that corresponds to what we just described. We are going to show that a multicontact vector field corresponding to the root  $\alpha$  cannot be identically zero in its components labeled by the intersection  $\mathcal{J}$  ( $\mathcal{S}_{\mu_j} \cap \mathcal{S}_{\mu_k}$ ). The reason of this lies in the fact that  $\mathcal{Y}(\alpha)$  contains a boundary simple root, namely  $\delta$ , and that  $\delta$  is adjacent to  $\mathcal{J}$ . Roughly speaking, the root  $\alpha$  is close enough to  $\mathcal{J}$ , and this allows a multicontact vector field corresponding to  $\alpha$  not to be killed before its coefficients arrive to the  $\mathcal{J}$ -positions.

In short, we prove that

$$(52) \quad \tau(E_\alpha^i)^{\mathcal{J}} \neq 0.$$

If the equation above holds, then the relation

$$\tau(E_\alpha^i)^{\mathcal{J}} = \tau(E_\alpha^j)^{\mathcal{J}}$$

forces  $F^j = \tau(E^j)^j$  to be non-zero because  $\tau(E_\alpha^j)^j \neq 0$ . On the other hand  $-\alpha \notin \mathcal{S}_{\mu_j}$ , for otherwise  $\delta$  would lie in  $\mathcal{S}_{\mu_j}$ . This implies that  $\tau(E_\alpha^j)^j$  is not in  $MC(N^{\mu_j})$  by Proposition 24. This, in turn, implies that  $F^{\mu_j}$ , hence  $F$ , is not a multicontact vector field, that is the contradiction we expected.

It remains to prove equation (52). Suppose  $\tau(E_\alpha^i)^{\mathcal{J}} = 0$ . This will give that  $\tau(E_{-\delta}^i)^{\mathcal{J}} = 0$  that, in turn, implies that  $\delta$  is not a boundary root, a contradiction. First, by  $\tau(E_\alpha^i)^{\mathcal{J}} = 0$ , it follows that for every  $E' \in \mathfrak{n}$  is

$$[\tau(E_\beta^i), \tau(E')] = \left[ \sum_{\gamma_1 \in \mathcal{J}^c} \sum_{i=1}^{m_{\gamma_1}} f_{\gamma_1, i} X_{\gamma_1, i}, \sum_{\gamma_2 \in \mathcal{J}} \sum_{j=1}^{m_{\gamma_2}} g_{\gamma_2, j} X_{\gamma_2, j} + \sum_{\gamma_3 \in \mathcal{J}^c} \sum_{k=1}^{m_{\gamma_3}} g_{\gamma_3, k} X_{\gamma_3, k} \right].$$

All terms of the bracket above lie in  $\mathfrak{X}(N^{\mathcal{J}^c})$ , except

$$f_{\gamma_1, i} X_{\gamma_1, i} (g_{\gamma_2, j}) X_{\gamma_2, j},$$

but  $X_{\gamma_1, i} (g_{\gamma_2, j}) = 0$ , for every  $\gamma_1 \in \mathcal{J}^c$  and  $\gamma_2 \in \mathcal{J}$ , because the coefficients  $g_{\gamma_2, j}$  are  $(\Sigma_+ \setminus \mathcal{J})$ -independent. Indeed,  $\mathcal{J}$  is a Hessenberg set and its complement  $\mathcal{J}^c$  defines an ideal  $\mathfrak{n}_{\mathcal{J}^c}$  in  $\mathfrak{n}$  whose normalizer contains  $\mathfrak{n}$ . Therefore, since  $E' \in \mathfrak{n}$ , it also lies in  $N_{\mathfrak{g}} \mathfrak{n}_{\mathcal{J}^c}$ , so that the coefficients of  $\tau(E')^{\mathcal{J}}$  are  $(\Sigma_+ \setminus \mathcal{J})$ -independent by Lemma 14. Hence  $[\tau(E_\alpha^i), \tau(E')] \in \mathfrak{X}(N^{\mathcal{J}^c})$ , that implies

$$\tau([E_\alpha^i, E'])^{\mathcal{J}} = [\tau(E_\alpha^i), \tau(E')]^{\mathcal{J}} = 0$$

for every  $E' \in \mathfrak{n}$ . The same argument can be iterated for showing that

$$(53) \quad \tau([[E_\alpha^i, E'], \dots, E^{(n)}])^{\mathcal{J}} = [[\tau(E_\alpha^i), \tau(E')], \dots, \tau(E^{(n)})]^{\mathcal{J}} = 0$$

for every collection of elements  $E', \dots, E^{(n)}$  in  $\mathfrak{n}$ .

Let  $\delta_1, \dots, \delta_q$  simple roots such that  $\alpha + \delta_1 + \dots + \delta_q = -\delta$  is a chain, and  $E_1 \in \mathfrak{g}_{\delta_1}, \dots, E_q \in \mathfrak{g}_{\delta_q}$  such that

$$[[E_\alpha^i, E_1], \dots, E_q] = E_{-\delta} \in \mathfrak{g}_{-\delta} \setminus \{0\}.$$

We apply (53) to the bracket above, there obtaining

$$\tau(E_{-\delta})^{\mathcal{J}} = 0.$$

Since  $\theta E_{-\delta} \in \mathfrak{n}$ , again the formula (53) together with Prop 6.52 in [21] gives

$$(54) \quad 0 = [\tau(E_{-\delta}), \tau(\theta E_{-\delta})]^{\mathcal{J}} = B(E_{-\delta}, \theta E_{-\delta})\tau(H_{\delta})^{\mathcal{J}}.$$

By Lemma 27, since  $\delta \in \mathcal{B}$ , there exists a simple root  $\delta' \in \mathcal{S}_{\mu_j}$  such that  $\delta + \delta'$  is a root. This implies that  $\langle \delta, \delta' \rangle \neq 0$ , because  $\delta - \delta'$  is never a root. Hence  $\delta'(H_{\delta}) \neq 0$ , so that  $H_{\delta} \in \mathfrak{g}^{\mu_j} \cap \mathfrak{g}^{\mu_k}$ . By Corollary 25, it follows that  $\tau(H_{\delta})^{\mathcal{J}} \neq 0$ , contradicting (54). This concludes our proof.  $\square$

**3.1. A remark on the group.** In this section we show that the group  $Q = \text{Int}(\mathfrak{q})$  acts on  $S$  via multicontact mappings. In order to do that, we define an alternative model for the Hessenberg manifolds, which is compatible with the stratified structure introduced in the first section.

Consider the group  $N_C = \exp \mathfrak{n}_C$ . Since  $\mathfrak{n}_C$  is an ideal in  $\mathfrak{n}$ , its exponential group is a normal subgroup of  $N$ . Therefore the quotient  $N/N_C$  is a nilpotent Lie group. We identify the Lie algebra of  $N/N_C$  with  $\mathfrak{n}/\mathfrak{n}_C$ , and we define a natural multicontact structure on this quotient simply considering the subbundles  $\{\langle \mathfrak{g}_{\delta} \rangle_{\mathfrak{n}_C} : \delta \in \Delta\}$ , where  $\langle E \rangle_{\mathfrak{n}_C}$  denotes the coset of  $E$  in  $\mathfrak{n}/\mathfrak{n}_C$ . Let  $f$  be a diffeomorphism between open subsets of  $N/N_C$ . Then  $f$  is a multicontact mapping if for every simple root  $\delta$

$$f_*(\langle \mathfrak{g}_{\delta} \rangle_{\mathfrak{n}_C}) \subset \langle \mathfrak{g}_{\delta} \rangle_{\mathfrak{n}_C}.$$

The coordinates system on the slice  $S$  define the analytic structure on  $N/N_C$ . Thus,  $N/N_C$  and  $S$  are diffeomorphic by the assignment

$$\chi : \langle (\{x_{\alpha,i}\}_{\alpha \in \Sigma_+}) \rangle_{N_C} \mapsto (\{x_{\alpha,i}\}_{\alpha \in \mathcal{R}}, 0),$$

where  $\langle n \rangle_{N_C}$  denotes the coset of  $n \in N$  in the quotient group. The differential  $\chi_*$  maps the left-invariant vector field  $\langle X_{\alpha,i} \rangle_{\mathfrak{n}_C}$  to  $\overline{X}_{\alpha,i}$ , therefore the multicontact structure on  $N/N_C$  is mapped onto the multicontact structure on  $S$ . The diffeomorphism  $\chi$  allows us to view  $S$ , and hence locally a Hessenberg manifold, as a nilpotent Lie group. From the point of view of multicontact mappings, we identify  $S$  with  $N/N_C$ .

Let  $Q = \text{Int}(\mathfrak{q})$ . We have  $\text{Int}(\mathfrak{q}) \subset \text{Int}(\mathfrak{g})$ , because  $\text{Int}(\mathfrak{q}) = e^{\text{ad}\mathfrak{q}}$ ,  $\text{Int}(\mathfrak{g}) = e^{\text{ad}\mathfrak{g}}$  and  $\mathfrak{q} \subset \mathfrak{g}$ .

LEMMA 29. *The action of every element  $q \in Q$  on  $N$  induces a well-posed action on the quotient  $N/N_C$ , namely*

$$\hat{q}(\langle n \rangle_{N_C}) = \langle [qn] \rangle_{N_C},$$

where  $[qn]$  is the  $N$ -component of  $qn$  in the Bruhat decomposition.

**PROOF.** Let  $n \in N$  and  $n_C \in N_C$ . Then  $n$  and  $nn_C$  both represent  $\langle n \rangle_{N_C} \in N/N_C$ . We show that  $[qn]$  and  $[qnn_C]$  represent the same element in  $N/N_C$ , that is  $[qn][qnn_C]^{-1} \in N_C$ . Let  $p \in P$  such that  $[qn] = qnp$ . Since  $N_C$  is a normal subgroup of  $Q$ , there exists  $n'_C \in N_C$  such that

$$\begin{aligned}[qnn_C] &= [n'_C qn] \\ &= n'_C [qn] \\ &= n'_C qnp.\end{aligned}$$

Then  $[qn][qnn_C]^{-1} = qnp(n'_C qnp)^{-1} = (n'_C)^{-1} \in N_C$ , as required.  $\square$

We prove the following proposition.

**PROPOSITION 30.** *Let  $Q$  be as above, and  $\mathcal{A}$  an open subset of  $N/N_C$ . For every  $q \in Q$ , the map*

$$\hat{q} : \mathcal{A} \subset N/N_C \rightarrow N/N_C$$

*is a multicontact mapping on  $\mathcal{A}$ . Furthermore  $\hat{q} = \text{id}_{\mathcal{A}}$  for every  $q \in N_C$ .*

**PROOF.** Since  $q \in G = \text{Int}(\mathfrak{g})$ , it is a multicontact mapping on  $G/P$ . Thus,  $q_*(\mathfrak{g}_\delta) \subseteq \mathfrak{g}_\delta$  for every simple root  $\delta$  (see Ch. 1, Sec. 2). Let  $E \in \mathfrak{g}_\delta$ , for some  $\delta \in \Delta$ , and consider a representative in  $\mathfrak{n}/\mathfrak{n}_C$  of  $\langle E \rangle_{\mathfrak{n}_C}$ , say  $E + E'$ , with  $E' \in \mathfrak{n}_C$ . Then

$$\hat{q}_*(\langle E \rangle_{\mathfrak{n}_C}) = \langle (l_q)_*(E + E') \rangle_{\mathfrak{n}_C}.$$

By definition

$$(l_q)_*(E') = \frac{d}{dt}(q \exp(tE')) \Big|_{t=0}.$$

Since  $[\mathfrak{q}, \mathfrak{n}_C] \subset \mathfrak{n}_C$ , a straightforward calculation implies that  $(l_q)_*(E') \in \mathfrak{n}_C$ . Therefore there exists  $E'' \in \mathfrak{n}_C$  such that

$$\hat{q}_*(\langle E \rangle_{\mathfrak{n}_C}) = \langle (l_q)_*(E + E') \rangle_{\mathfrak{n}_C} = \langle q_*(E) + E'' \rangle_{\mathfrak{n}_C} \subset \langle \mathfrak{g}_\delta + \mathfrak{n}_C \rangle_{\mathfrak{n}_C} \subset \langle \mathfrak{g}_\delta \rangle_{\mathfrak{n}_C}.$$

Since  $\langle n_C n \rangle_{N_C} = \langle n \rangle_{N_C}$ , it follows that  $\hat{q}_*$  maps  $\langle n \rangle_{N_C}$  in itself, for every  $n \in N$ . Hence the proposition holds.  $\square$

The above proposition, together with the diffeomorphism  $\chi$ , tell us that  $Q/N_C$  is a group of multicontact mappings on  $S$ . Indeed, for every  $q \in Q$ , the map  $\chi \hat{q} \chi^{-1}$  is multicontact on  $S$ , and it coincides with the identity whenever  $q \in N_C$ . This fact implies in particular that  $\mathfrak{q}/\mathfrak{n}_C$  is a Lie algebra of multicontact vector fields, so that we obtain another proof of Theorem 20.

**3.2. The case of  $A_l$ .** We conclude this section observing that hypothesis (I) of Theorem 22 always holds if we consider Hessenberg subsets in the root system  $A_l$ . The underlying vector space of  $A_l$  is  $V = \{v \in \mathbb{R}^{l+1} : \langle v, e_1 + \dots + e_{l+1} \rangle = 0\}$ , and the roots are  $A_l = \{e_i - e_j : i \neq j\}$ , where we choose the positive ones fixing  $\Sigma_+ = \{e_i - e_j : i < j\}$ . A basis of simple roots is given by  $\Delta = \{\delta_1 = e_1 - e_2, \delta_2 = e_2 - e_3, \dots, \delta_l = e_l - e_{l+1}\}$ , and the highest root with respect to this basis is  $\omega = \delta_1 + \dots + \delta_l$ . In [6], [7], [8] the author gives a classification of simple Lie algebras. In particular, by [7] we desume that the restricted root spaces associated to a simple Lie algebra with root system  $A_l$  have the same dimension, that is all roots have the same multiplicity, and the classification is the following:

- (1)  $m_\omega = 1$ , that corresponds to  $(\mathfrak{sl}(l+1, \mathbb{R}), \mathfrak{so}(l+1))$ ;
- (2)  $m_\omega = 2$ , that corresponds to  $(\mathfrak{sl}(l+1, \mathbb{C}), \mathfrak{su}(l+1))$ ;
- (3)  $m_\omega = 4$ , that corresponds to  $(\mathfrak{sl}(l+1, \mathbb{H}), \mathfrak{sp}(l+1))$ ;
- (4)  $m_\omega = 8$ , possible only for  $l = 2$ , it corresponds to  $(\mathfrak{e}_{(6,-26)}, \mathfrak{f}_4)$ .

If  $l = 2$ , the Hessenberg proper subsets in  $A_l$  are  $\mathcal{R} = \{\delta_1\}$ ,  $\mathcal{R} = \{\delta_2\}$ ,  $\mathcal{R} = \{\delta_1, \delta_2\}$ . In all these cases the study of multicontact mappings reduces to the case of rank one nilpotent Iwasawa Lie algebras. Because of hypothesis (II) of Theorem 22 and the remark thereafter, we restrict ourselves to the case  $l > 2$ . We have a rather obvious consequence.

**PROPOSITION 31.** *Let  $\mathfrak{g}$  a simple Lie algebra with root system  $A_l$ ,  $l > 2$ . Let  $\mathcal{R} \subseteq \Sigma_+$  a subset of Hessenberg type. Then each shadow  $\mathcal{S}_\mu$  define an Iwasawa nilpotent subalgebra of  $\mathfrak{n}$  of the following form*

$$\mathfrak{n}' = \bigoplus_{\alpha \in \mathcal{S}_\mu} \mathfrak{g}_\alpha.$$

**PROOF.** Let  $\mu \in \mathcal{R}_M$ . Then  $\mu = \delta_i + \delta_{i+1} + \dots + \delta_{i+h}$  for some  $i \in \{1, \dots, l-1\}$ , and it is the highest root of the root system generated by  $\mathcal{Y}(\mu) = \{\delta_i, \dots, \delta_{i+h}\}$ . Indeed, any root chain in  $A_l$  allows a simple root to appear at most once. Therefore, the Iwasawa subalgebra generated by  $\mathcal{Y}(\mu)$  coincides with

$$\mathfrak{n}' = \bigoplus_{\alpha \in \mathcal{S}_\mu} \mathfrak{g}_\alpha,$$

as required.  $\square$

#### 4. A counter-example

In this section we show with an example that the converse of Theorem 22 fails when we remove hypothesis (I). More precisely, this example suggests

a general conjecture for the Lie algebra of multicontact vector fields on a Hessenberg manifold, that reduces to coincide with  $\mathfrak{q}/\mathfrak{n}_C$  in the case that (I) holds.

Consider the simple Lie group

$$\mathrm{Sp}(2, \mathbb{R}) = \{A \in \mathrm{SL}(2n, \mathbb{R}) : A^t JA = J\},$$

where

$$J = \begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix},$$

and  $I_2$  is the identity in  $\mathrm{GL}(2, \mathbb{R})$ . Take its Lie algebra

$$\mathfrak{sp}(2, \mathbb{R}) = \{A \in \mathfrak{gl}(2n, \mathbb{R}) : A^t J + JX = 0\}.$$

Denote by  $H_{s,t} = \mathrm{diag}(s, t, -s, -t)$ . Then the Cartan space  $\mathfrak{a}$  is defined by  $\{H_{s,t} : s, t \in \mathbb{R}\}$ . The standard restricted simple roots are  $\alpha$  and  $\beta$ , where  $\alpha(H_{s,t}) = t - s$  and  $\beta(H_{s,t}) = -2t$ . Moreover, the set of the positive roots is  $\Sigma_+ = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta\}$ . The Iwasawa subalgebra  $\mathfrak{n}$  of  $\mathfrak{sp}(2, \mathbb{R})$  is the direct sum of the restricted root spaces corresponding to the roots in  $\Sigma_+$ , namely  $\mathfrak{n} = \mathrm{span}\{E_U, E_X, E_Y, E_Z\}$ , where

$$E_U = \begin{bmatrix} 0 & 0 \\ -\frac{1}{2} & 0 \\ & 0 & \frac{1}{2} \\ & 0 & 0 \end{bmatrix} \quad E_X = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 2 \end{bmatrix},$$

$$E_Y = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \quad E_Z = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

We fix coordinates on  $N = \exp \mathfrak{n}$ , the nilpotent Iwasawa subgroup of  $\mathrm{Sp}(2, \mathbb{R})$ . Any element  $n$  in  $N$  consists of a  $4 \times 4$  matrix that we write as

$$n(u, x, y, z) = \begin{bmatrix} 1 & -\frac{1}{2}u & z - \frac{1}{2}uy & y - ux \\ 0 & 1 & y & 2x \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2}u & 1 \end{bmatrix}.$$

In other words, we consider  $\mathbb{R}^4$  with the group law:

$$n(u, x, y, z)n(u', x', y', z') = \bar{n}(u+u', x+x', y+y'+ux', z+z'+uy'+\frac{1}{2}u^2x').$$

A basis of left-invariant vector fields for the Lie algebra  $\mathfrak{n}$  is given by

$$\begin{aligned} U &= \frac{\partial}{\partial u}; \\ X &= \frac{\partial}{\partial x} + u \frac{\partial}{\partial y} + \frac{u^2}{2} \frac{\partial}{\partial z}; \\ Y &= \frac{\partial}{\partial y} + u \frac{\partial}{\partial z}; \\ Z &= \frac{\partial}{\partial z}, \end{aligned}$$

with brackets

$$[U, X] = Y, \quad [U, Y] = Z, \quad [X, Y] = [X, Z] = [Y, Z] = 0.$$

Take a regular element  $H_0$  in  $\mathfrak{a}$  and fix  $\mathcal{R} = \{\alpha, \beta, \alpha+\beta\}$ . In order to study the multicontact vector fields on the Hessenberg manifold that corresponds to these data, we consider the slice  $S$ , defined by the equation  $z = 0$ . A generating set of vector fields on  $S$  is

$$\begin{aligned} \overline{U} &= \frac{\partial}{\partial u}; \\ \overline{X} &= \frac{\partial}{\partial x} + u \frac{\partial}{\partial y}; \\ \overline{Y} &= \frac{\partial}{\partial y}, \end{aligned}$$

where  $[\overline{U}, \overline{X}] = \overline{Y}$  is the only non-zero bracket.

Consider a vector field on an open set  $\mathcal{A} \subseteq S$ , namely  $F = f_u \overline{U} + f_x \overline{X} + f_y \overline{Y}$ , where  $f_u, f_x$  and  $f_y$  are smooth functions on  $\mathcal{A}$ . Then  $F$  is a multicontact vector field on  $\mathcal{A}$  if and only if

$$\begin{aligned} [F, \overline{U}] &= \alpha \overline{U}, \\ [F, \overline{X}] &= \beta \overline{X}, \end{aligned}$$

for some functions  $\alpha$  and  $\beta$  on  $\mathcal{A}$ . The two conditions give rise to the differential equations

$$\begin{cases} f_x + \overline{U}f_y = 0 \\ \overline{U}f_x = 0 \end{cases} \quad \begin{cases} -f_u + \overline{X}f_y = 0 \\ \overline{X}f_u = 0 \end{cases}$$

that imply

$$(55) \quad \begin{cases} \overline{X}^2 f_y = 0 \\ \overline{U}^2 f_y = 0. \end{cases}$$

Notice that, since  $\mathcal{R}$  is defined as a single shadow, we obtain only one model-system, so that no cross-conditions appear. Moreover, the system above coincides with the multicontact conditions for the coefficients of a vector field on the nilpotent Iwasawa subgroup of  $\mathrm{SL}(3, \mathbb{R})$  (see (8), Ch. 1). Therefore,

$$MC(S) \cong \mathfrak{sl}(3, \mathbb{R}).$$

We show that this result does not coincide with the claim of Theorem 22, namely  $\mathfrak{sl}(3, \mathbb{R})$  is not isomorphic to  $\mathfrak{q}/\mathfrak{n}_C$ . Indeed, in the present example  $C = \{\omega = 2\alpha + \beta\}$ , and  $\mathfrak{n}_C = \mathfrak{g}_\omega$ . Hence, the normalizer in  $\mathfrak{sp}(2, \mathbb{R})$  of  $\mathfrak{n}_C$  is

$$(56) \quad \mathfrak{q} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta \oplus \mathfrak{g}_{\alpha+\beta} \oplus \mathfrak{g}_\omega \oplus \mathfrak{a} \oplus \mathfrak{g}_{-\beta}.$$

Thus

$$\dim(\mathfrak{q}/\mathfrak{n}_C) = \dim(\mathfrak{q}) - \dim(\mathfrak{n}_C) = 6,$$

whereas the dimension of  $\mathfrak{sl}(3, \mathbb{R})$  is 8.

We want to interpret the result that we obtained for  $MC(S)$  in a more general context. To this end, first we identify  $\mathfrak{sl}(3, \mathbb{R})$  with a subspace of  $\mathfrak{sp}(2, \mathbb{R})$ , and secondly we interpret this subspace in terms of the Hessenberg data.

We consider the multicontact vector fields on  $N$ , and we see how to interpret them as multicontact vector fields on  $S$ . If  $V = v_u U + v_x X + v_y Y + v_z Z$  is a multicontact vector field on  $N$ , then it has the form  $\tau(E)$  for some  $E \in \mathfrak{sp}(2, \mathbb{R})$ , and its coefficients are polynomials. Decompose  $\mathfrak{sp}(2, \mathbb{R})$  according to its root space decomposition, and choose a basis corresponding to the roots. By [11], any root space determines a unique (up to a real factor) multicontact vector field that, in turn, defines a polynomial  $v_z$ , namely

$$\begin{aligned} \mathfrak{g}_\omega &\leftrightarrow 1 & \mathfrak{g}_{-\omega} &\leftrightarrow (uy - 2z)^2 \\ \mathfrak{g}_{\alpha+\beta} &\leftrightarrow u & \mathfrak{g}_{-\alpha-\beta} &\leftrightarrow (y - ux)(uy - 2z) \\ \mathfrak{g}_\beta &\leftrightarrow u^2/2 & \mathfrak{g}_{-\beta} &\leftrightarrow (y - ux)^2/2 \\ \mathfrak{g}_\alpha &\leftrightarrow (y - ux) & \mathfrak{g}_{-\alpha} &\leftrightarrow u(uy - 2z) \\ \mathfrak{g}_0 = \mathfrak{a} &\leftrightarrow \{(uy - 2z), u(y - ux)\}. \end{aligned}$$

We now define a linear map  $D$  between the algebra of polynomials associated with  $MC(N)$  and the algebra of polynomials associated with  $MC(S)$ . We determine  $D$  by extending linearly the assignment

$$D : v_z(u, x, y, z) \mapsto (Uv_z)(u, x, y, uy/2).$$

Consider the image of a basis of polynomials corresponding to the root spaces:

$$\begin{aligned} D(1) &= 0 & D((uy - 2z)^2) &= 0 \\ D(u) &= 1 & D((y - ux)(uy - 2z)) &= y(y - ux) \\ D(u^2/2) &= u & D((y - ux)^2) &= -x(y - ux) \\ D((y - ux)) &= -x & D(u(uy - 2z)) &= uy \\ D((uy - 2z)) &= y & D(u(y - ux)) &= y - 2ux. \end{aligned}$$

The image of  $D$  gives a set of linear independent polynomials all solving (55). Therefore, by dimensional consideration, they generate  $MC(S)$ . Recalling that  $MC(S) \cong \mathfrak{sl}(3, \mathbb{R})$ , it follows that  $D$  establishes in turn a vector space isomorphism between the subspace

$$(57) \quad \mathfrak{a} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta \oplus \mathfrak{g}_{\alpha+\beta} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-\beta} \oplus \mathfrak{g}_{-\alpha-\beta}$$

of  $\mathfrak{sp}(2, \mathbb{R})$  and  $\mathfrak{sl}(3, \mathbb{R})$ .

We still make a further step, that allows us to define (57), and hence  $MC(S)$ , by the Hessenberg data, in a form that can be viewed as a generalization of the characterization given in Theorem 22. Let  $\mathcal{D} = \{\gamma \in \Sigma_- : \mathfrak{g}_\gamma \subset \mathfrak{q}\}$  and  $\mathcal{I} = \cap_{\mu \in \mathcal{R}_M} \mathcal{S}_\mu$ . Write

$$\begin{aligned} \mathfrak{q} \cap \bar{\mathfrak{n}} &= \sum_{\gamma \in \mathcal{D}} \mathfrak{g}_\gamma, \\ \bar{\mathfrak{n}}^\mathcal{I} &= \sum_{\gamma \in -\mathcal{I}} \mathfrak{g}_\gamma, \\ \mathfrak{b} &= \sum_{\gamma \in \mathcal{D} \cup (-\mathcal{I})} \mathfrak{g}_\gamma. \end{aligned}$$

Notice that  $(\bar{\mathfrak{n}}^\mathcal{I} \setminus (\mathfrak{q} \cap \bar{\mathfrak{n}}))$  is isomorphic as a vector space to  $\bar{\mathfrak{n}}/\mathfrak{b}$ . In the case study  $\mathcal{D} = \{-\beta\}$  and  $\mathcal{I} = \{\alpha, \beta, \alpha + \beta\}$ . We have the following vector space identifications:

$$\mathfrak{a} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta \oplus \mathfrak{g}_{\alpha+\beta} \oplus \mathfrak{g}_{-\beta} \cong \mathfrak{q}/\mathfrak{n}_C,$$

and

$$\mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-\alpha-\beta} \cong \bar{\mathfrak{n}}/\mathfrak{b}.$$

Hence

$$(58) \quad MC(S) \cong \mathfrak{q}/\mathfrak{n}_C \oplus \overline{\mathfrak{n}}/\mathfrak{b}.$$

We conjecture that identification (58) is true in general settings.

We still make a last remark. If (I) of Theorem 22 holds, the space  $\overline{\mathfrak{n}}^T$  is a subset of  $\mathfrak{q}$  (this follows from Lemma 27), so that the second term on the right hand in the direct sum (58) vanishes, thereby obtaining Theorem 22. In the case study,  $\mathfrak{n}/\mathfrak{n}_C$  is the Iwasawa algebra whose underlying root system is of type  $A_2$ . Hence,  $\mathfrak{n}/\mathfrak{n}_C$  is isomorphic to the nilpotent Iwasawa subalgebra of  $\mathfrak{sl}(3, \mathbb{R})$ . Generalizing, we can say that each shadow  $\mathcal{S}_\mu$  generates a vector space that, viewed as the quotient  $\mathfrak{n}/\mathfrak{n}_{\mathcal{S}_\mu^c}$ , with  $\mathcal{S}_\mu^c = \Sigma_+ \setminus \mathcal{S}_\mu$ , inherits from  $\mathfrak{n}$  the stratified structure of an Iwasawa nilpotent Lie algebra. In this context, we believe that the proof of (58) is an adaptation of the proof given for Theorem 22, and the most effort seems to be the characterization of the normalizer  $\mathfrak{q}$  in terms of roots, that is an analogous of Lemma 27.



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